RIGIDITY OF CONFIGURATIONS OF POINTS AND SPHERES

Opal Graham
Florida State University

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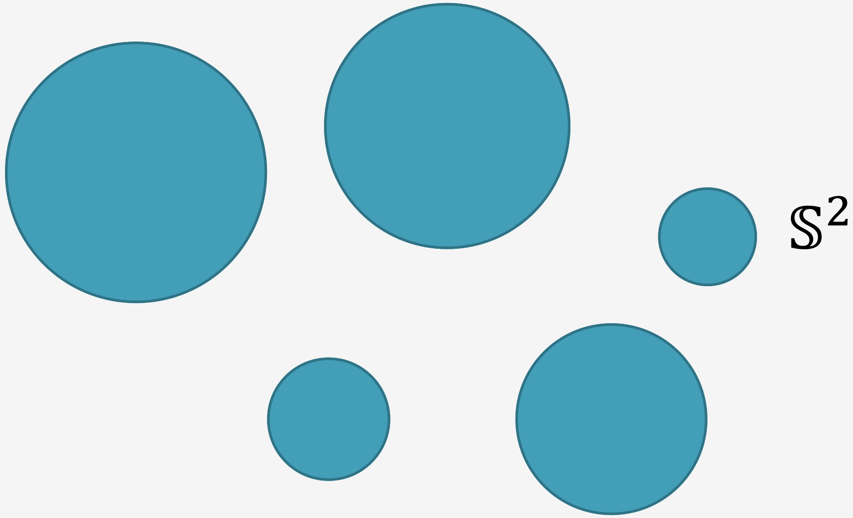
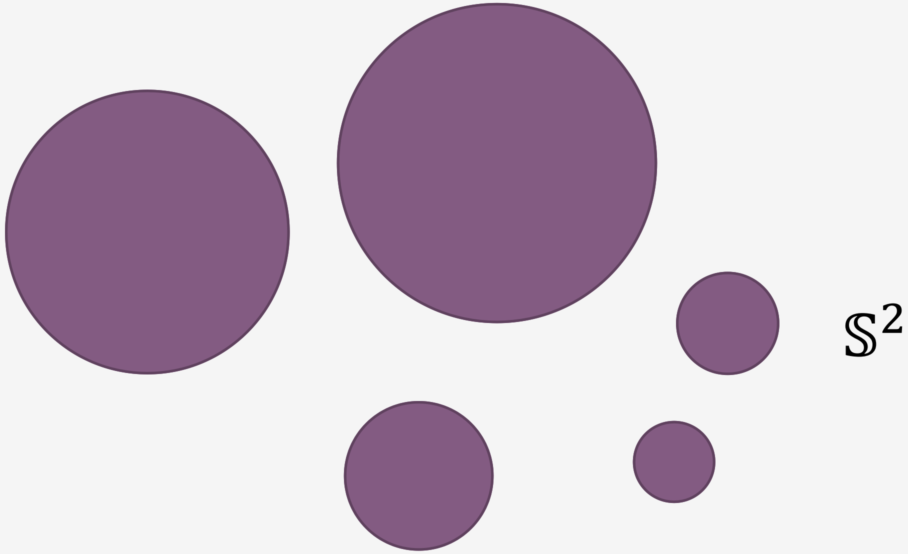
Mathematics

Beardon & Minda

Conformal automorphisms of finitely connected regions ('08)





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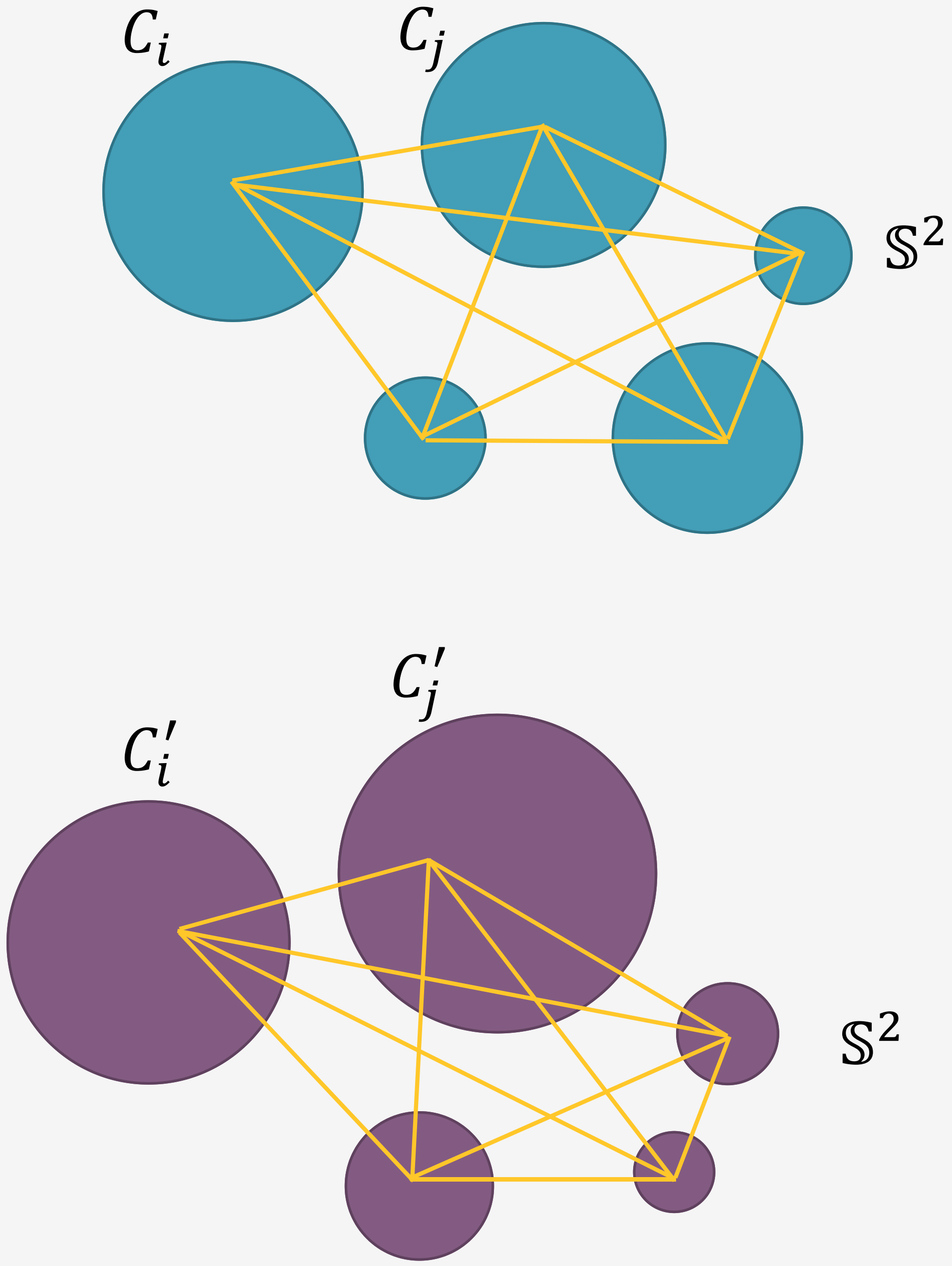

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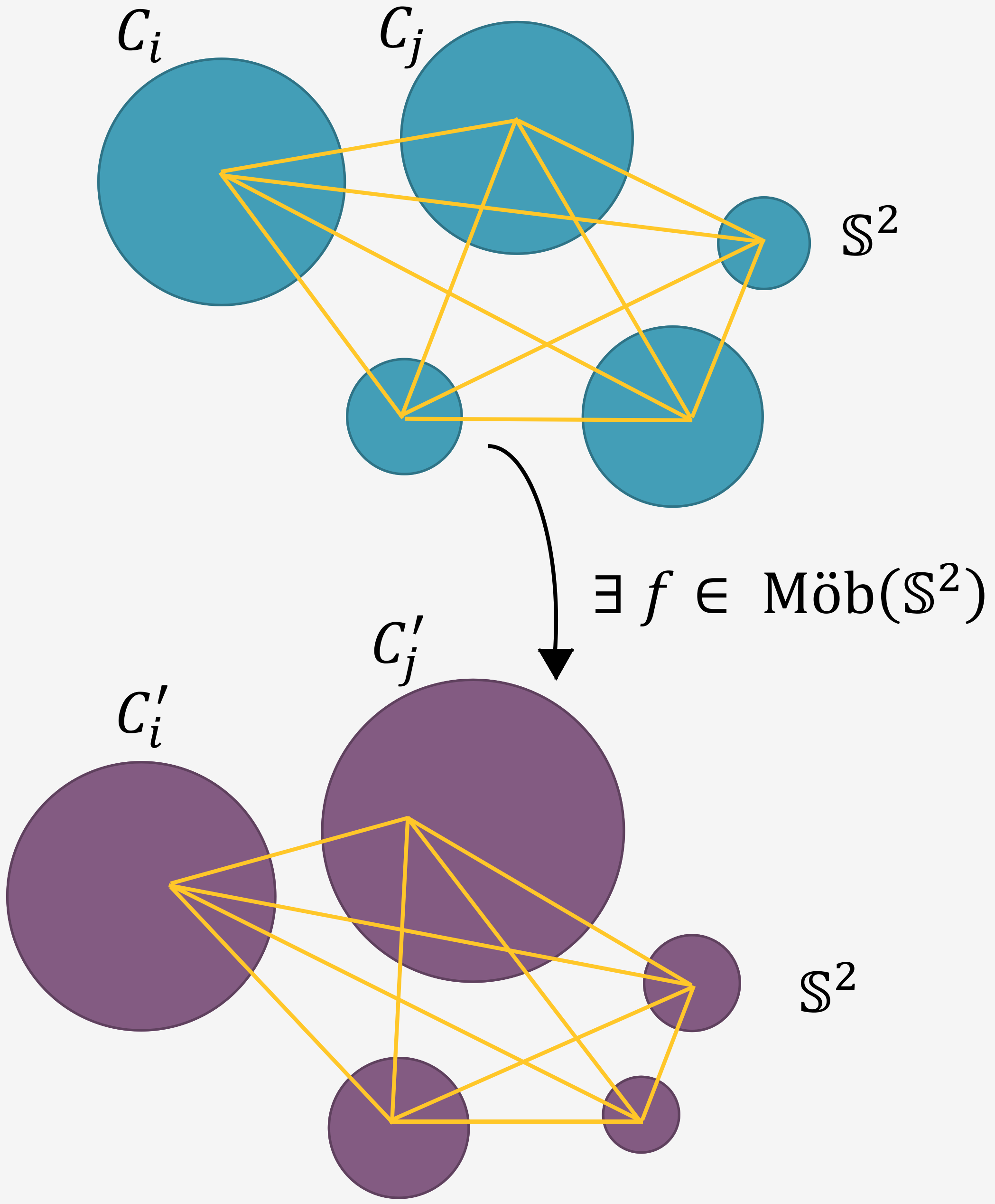

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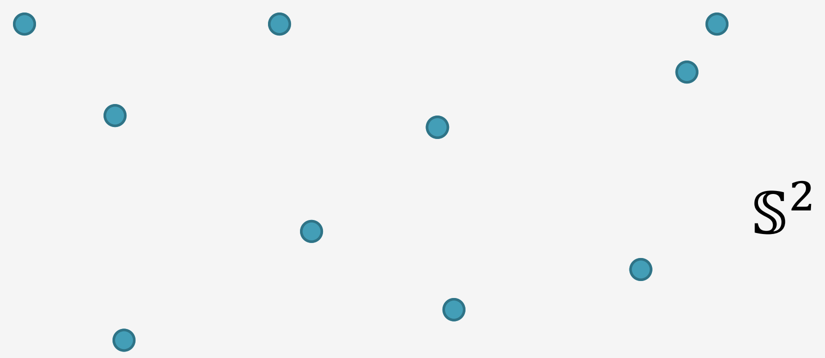

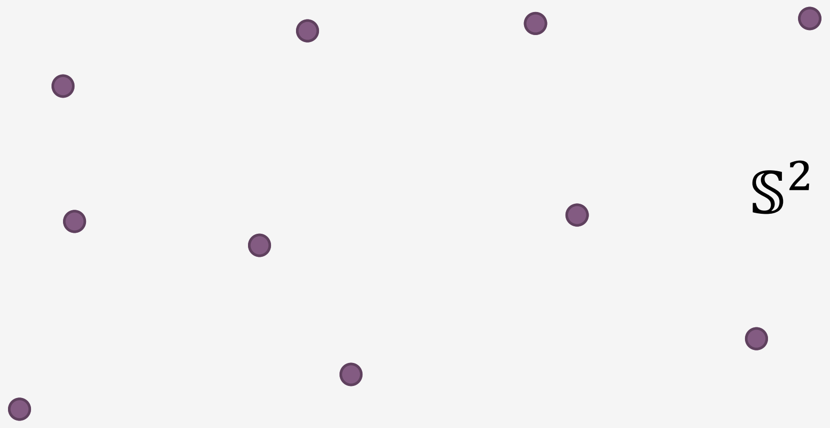

$$|p_i, p_j, p_k, p_l| = |p_i', p_j', p_k', p_l'| \ \forall \text{ 4-tuples}$$

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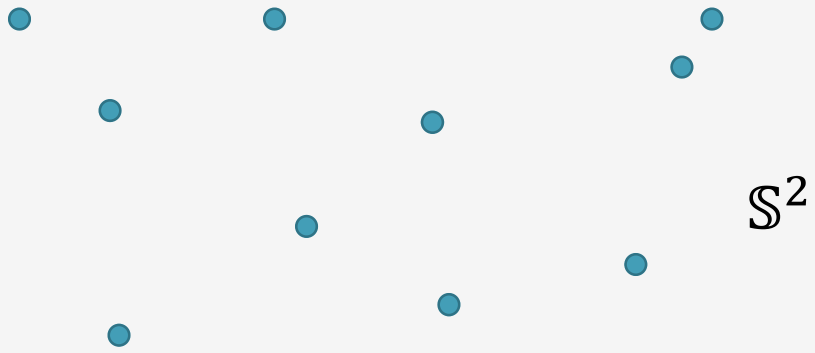

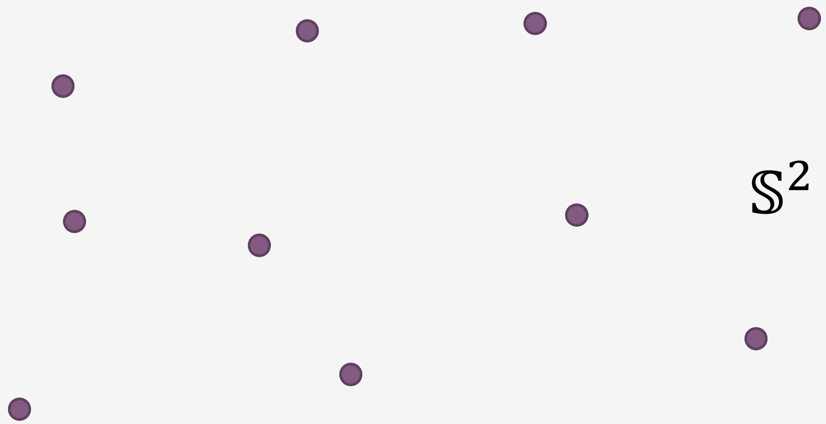

$$\left|p_i, p_j, p_k, p_l\right| = \left|p'_i, p'_j, p'_k, p'_l\right| \; \forall \text{ 4-tuples}$$

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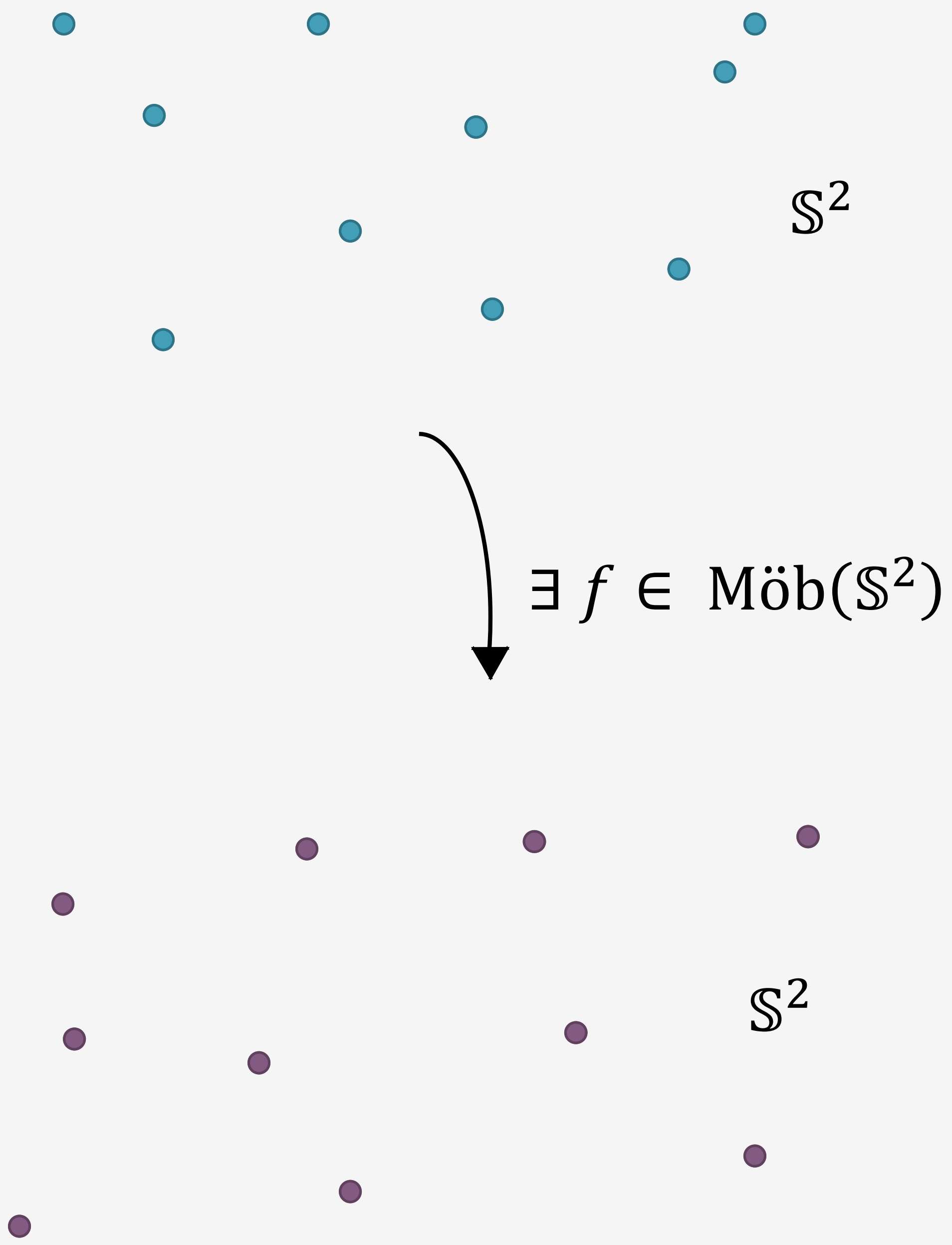
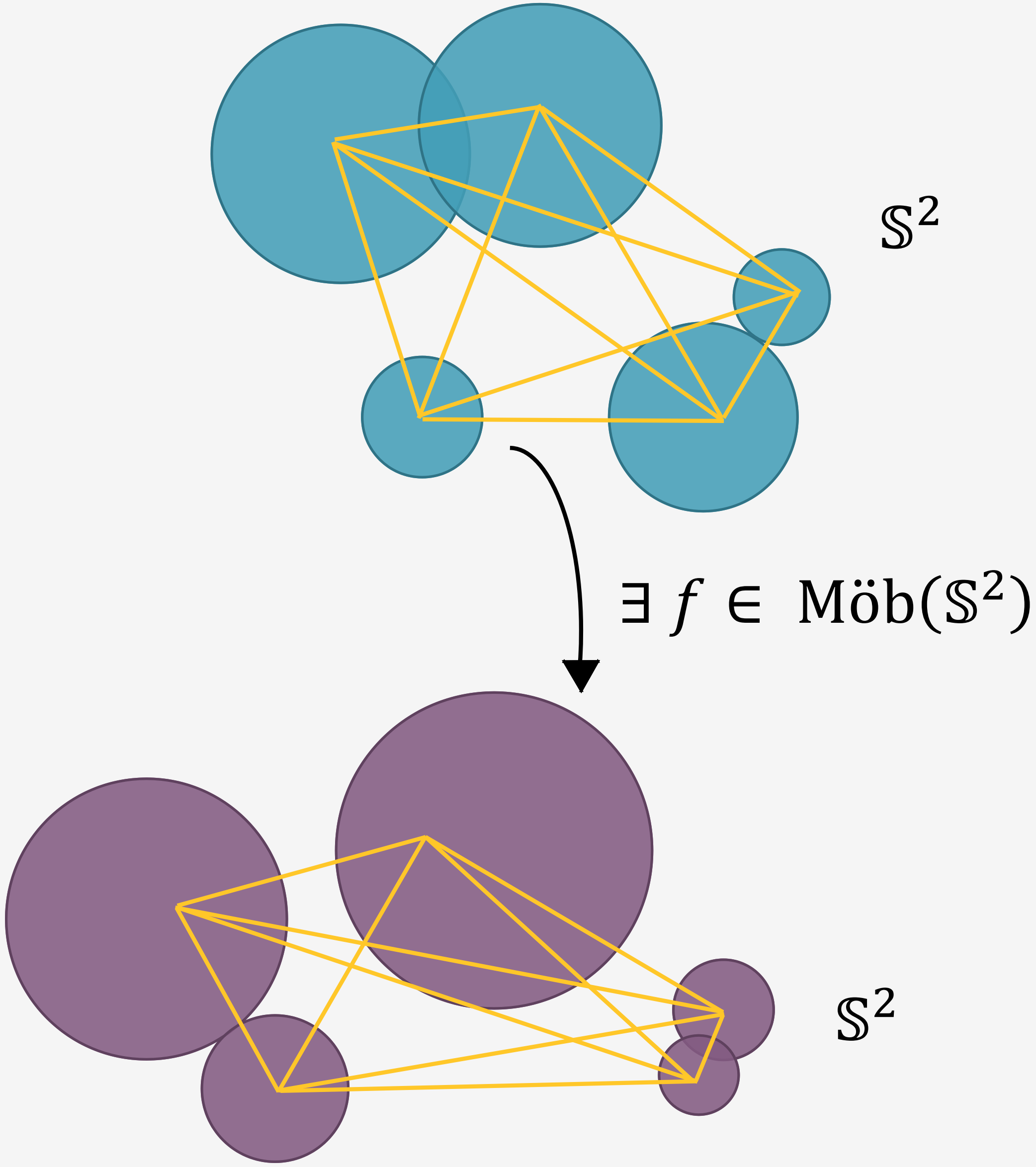


Question 1. Is the conclusion valid when $C_1, \dots, C_m$ are any set of m distinct circles in $\mathbb{C}_\infty$?



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Conformal automorphisms of finitely connected regions ('08)

Question 1. Is the conclusion valid when $C_1, \dots, C_m$ are any set of m distinct circles in $\mathbb{C}_\infty$?

Crane & Short Answer: Yes, with appropriate conditions.

Beardon & Minda

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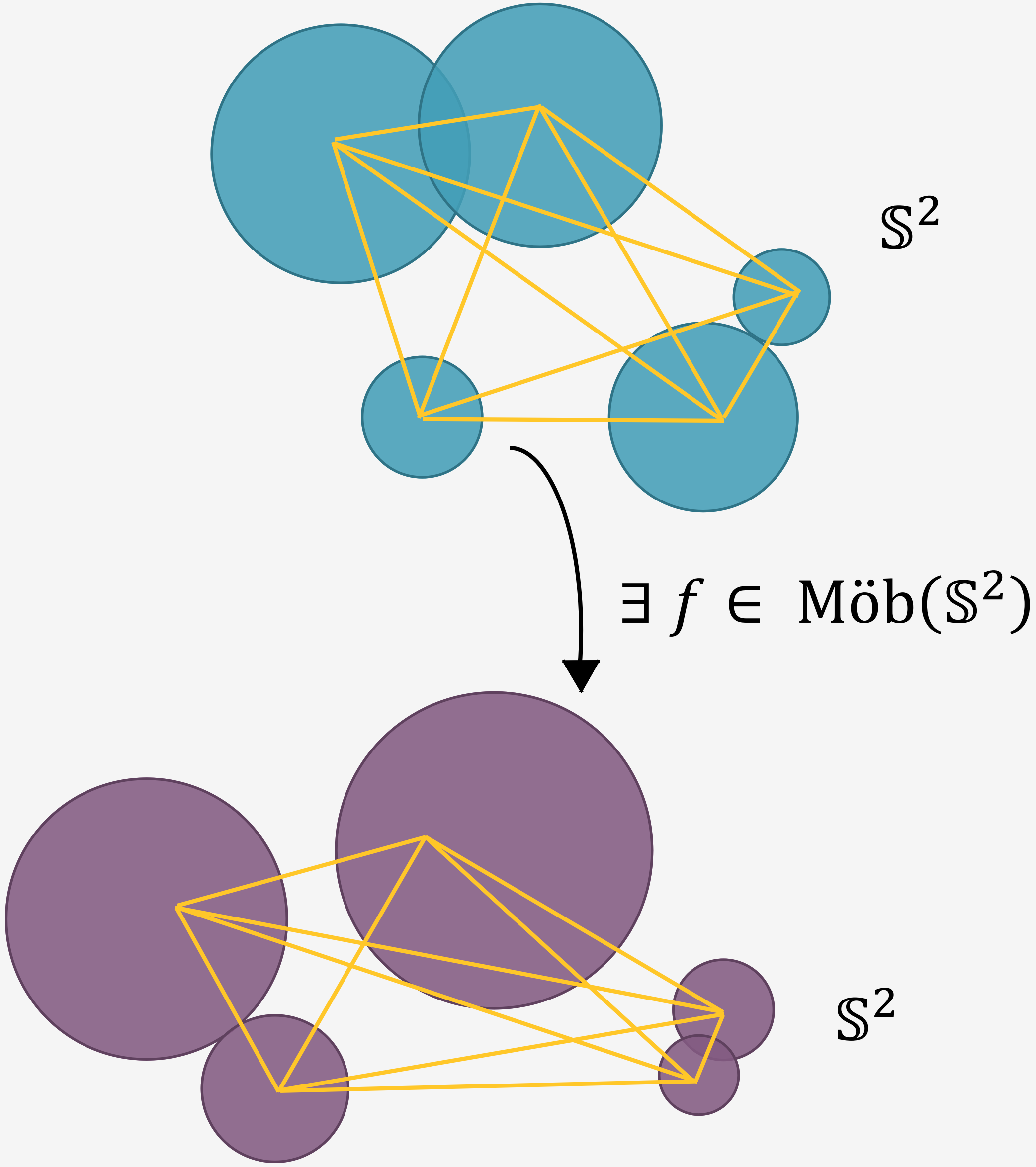

Question 2. Do both theorems generalize to higher dimensions?

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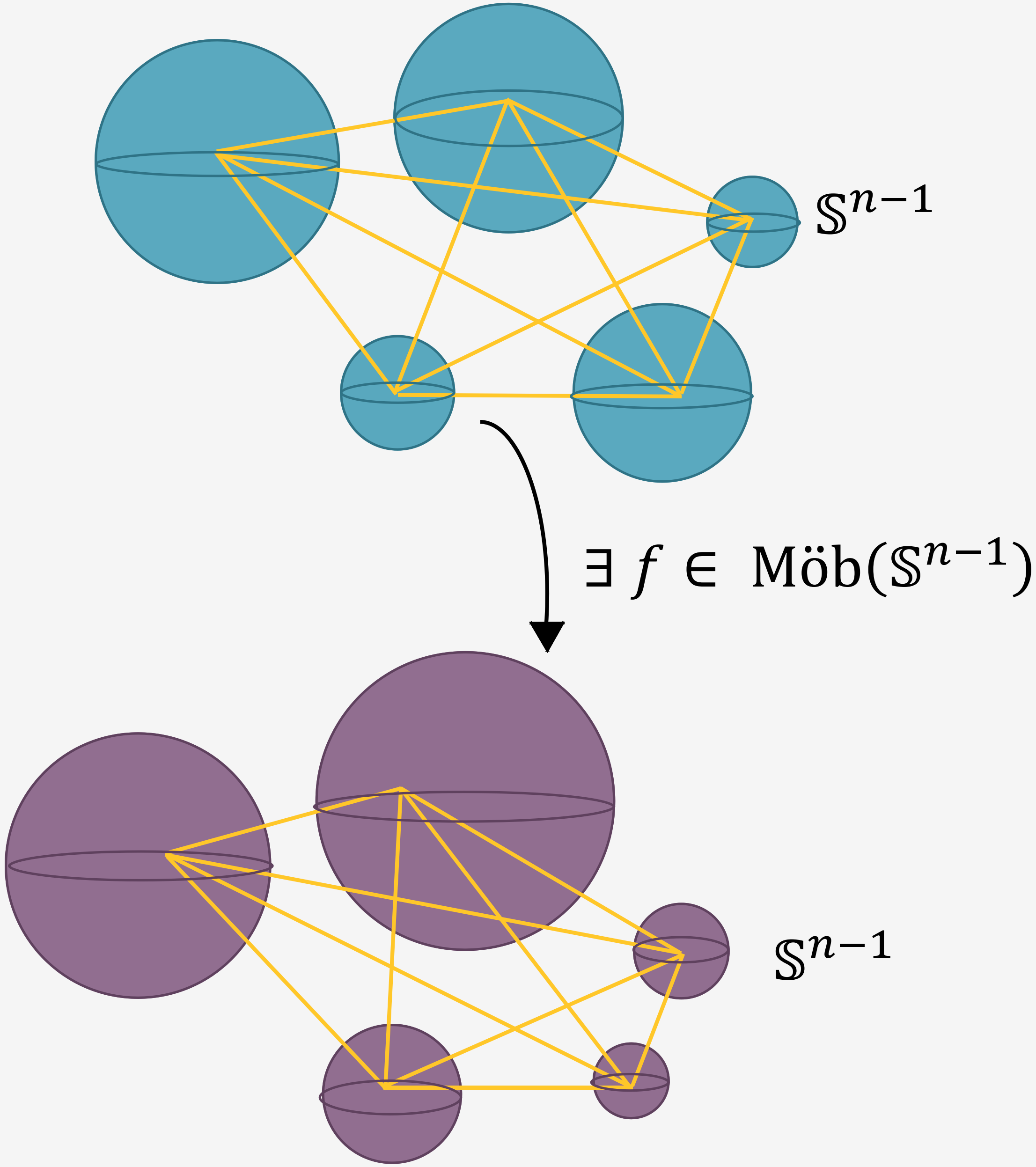

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Crane & Short Answer:
Yes

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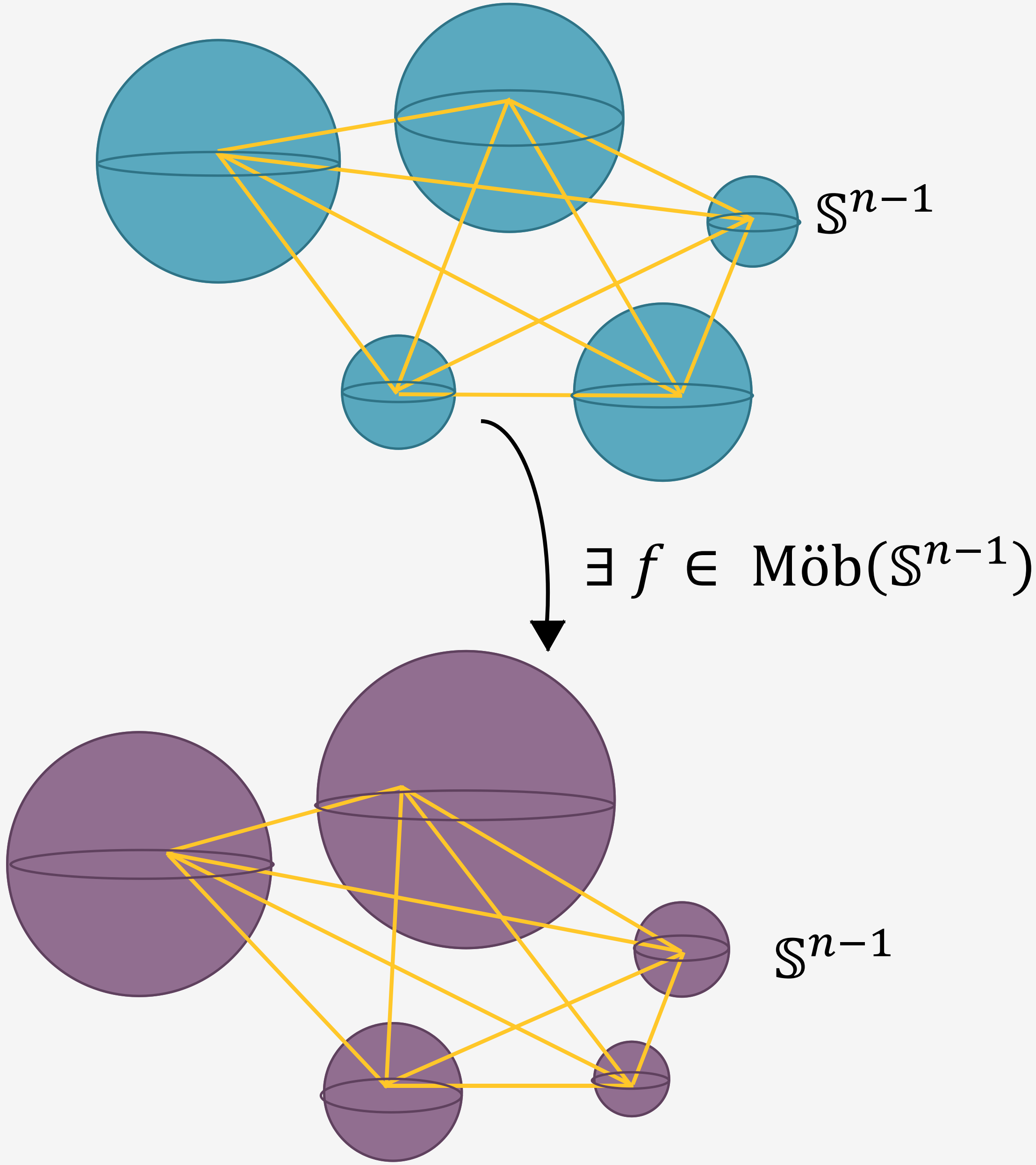

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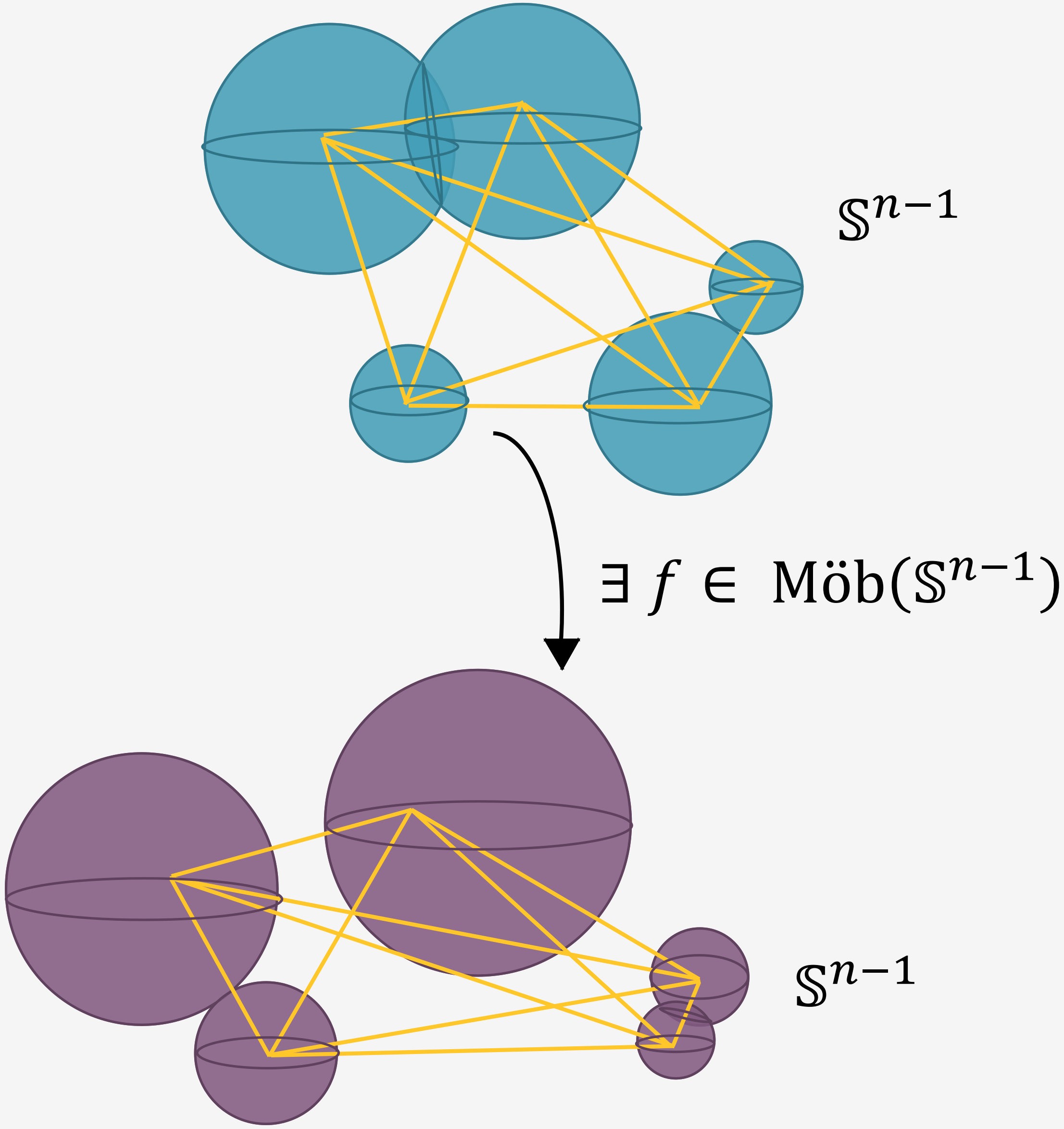
Crane & Short

Rigidity of Configurations of Balls and Points in the N-Sphere ('11)



Requires *all* inversive distance information.

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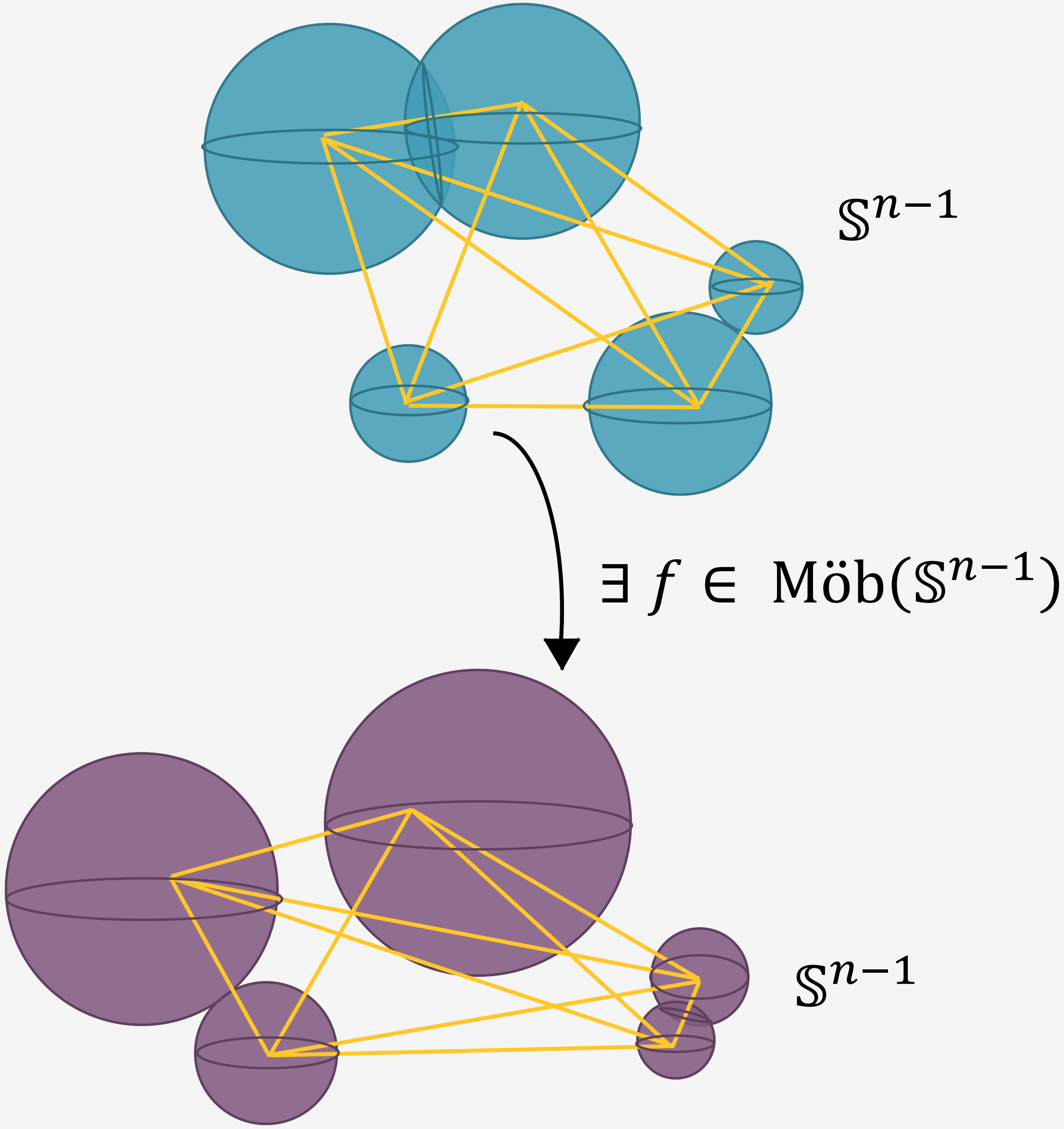

Requires *all* inversive distance information.

New question: Can these results be improved upon?

Crane & Short

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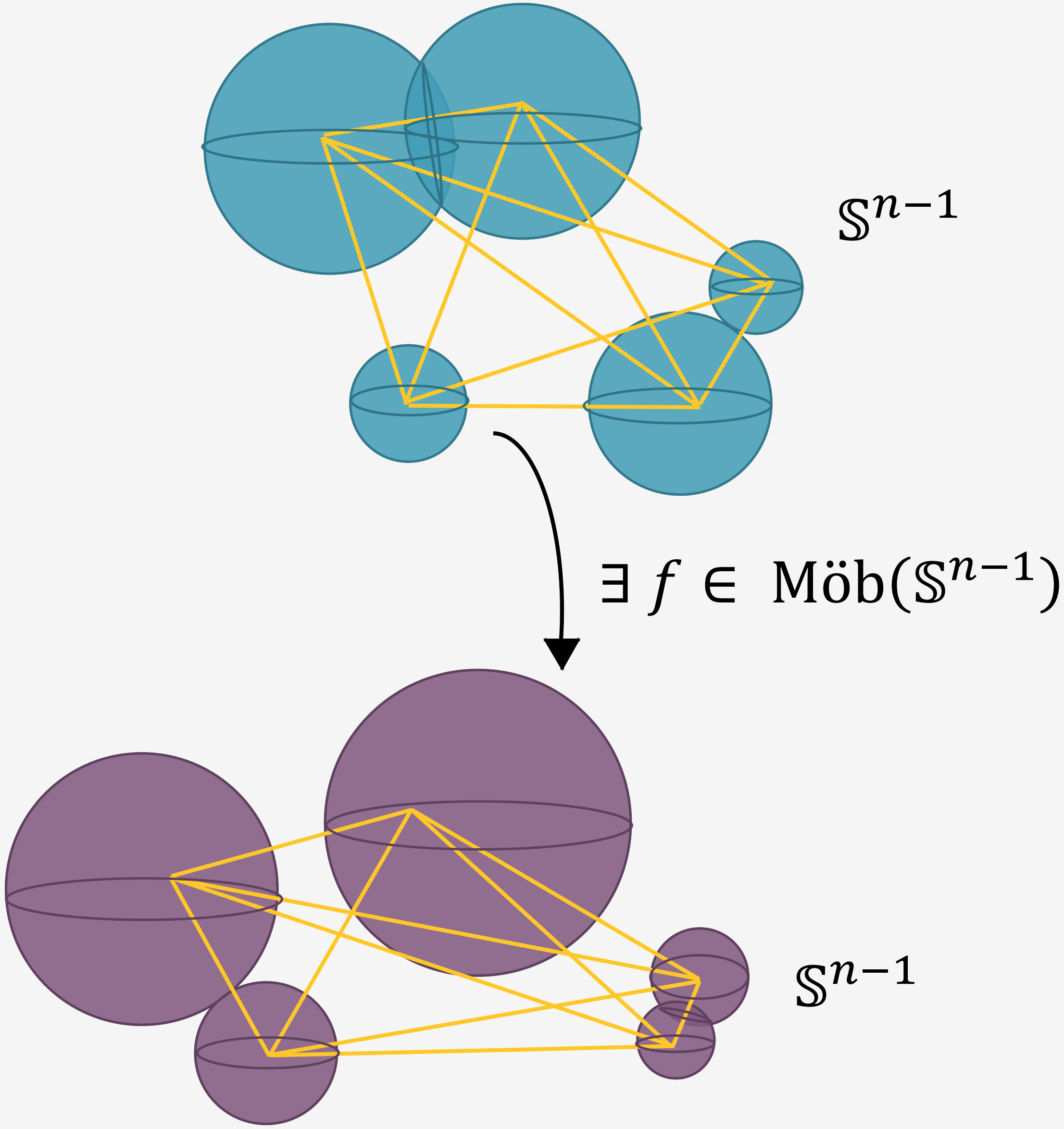
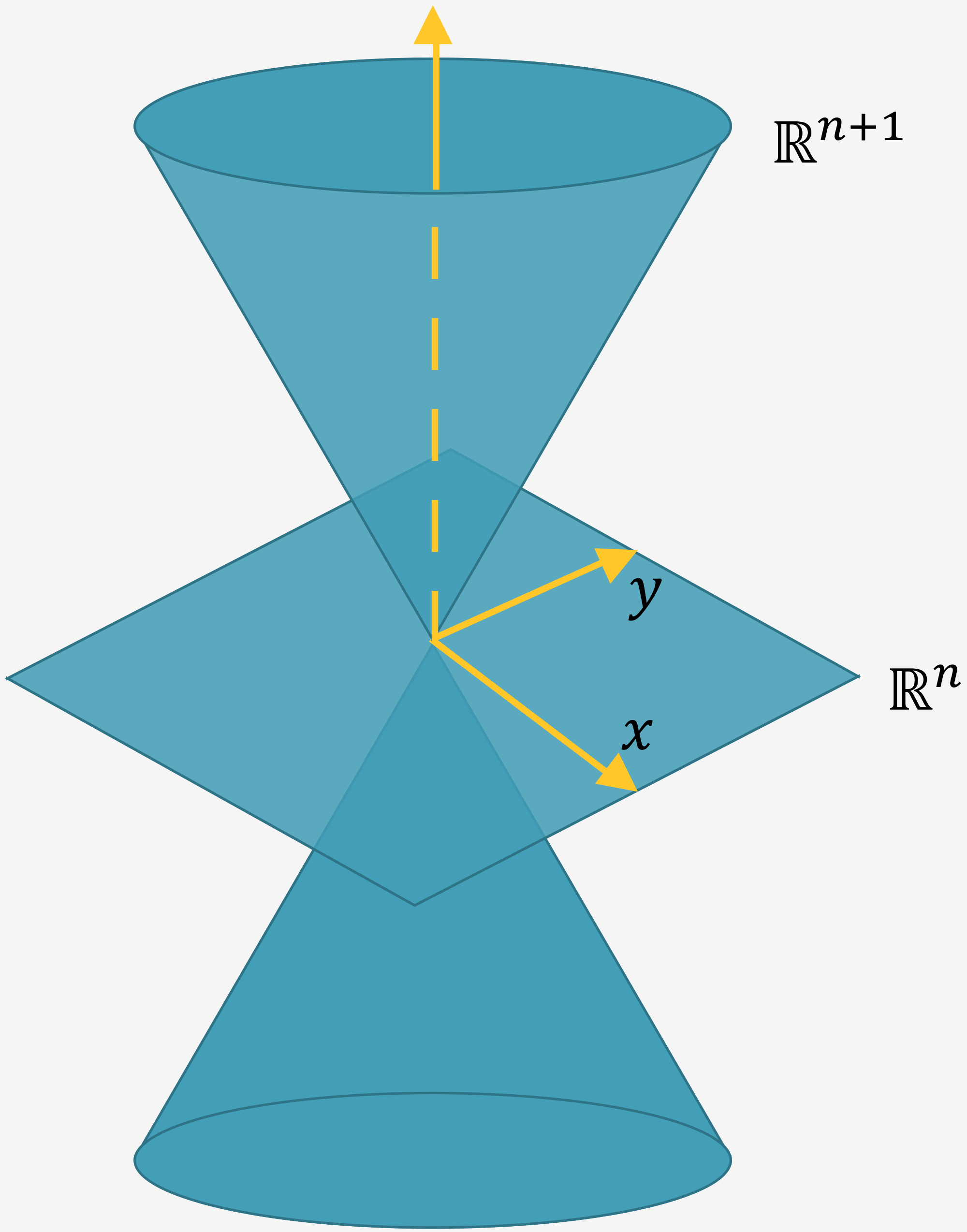


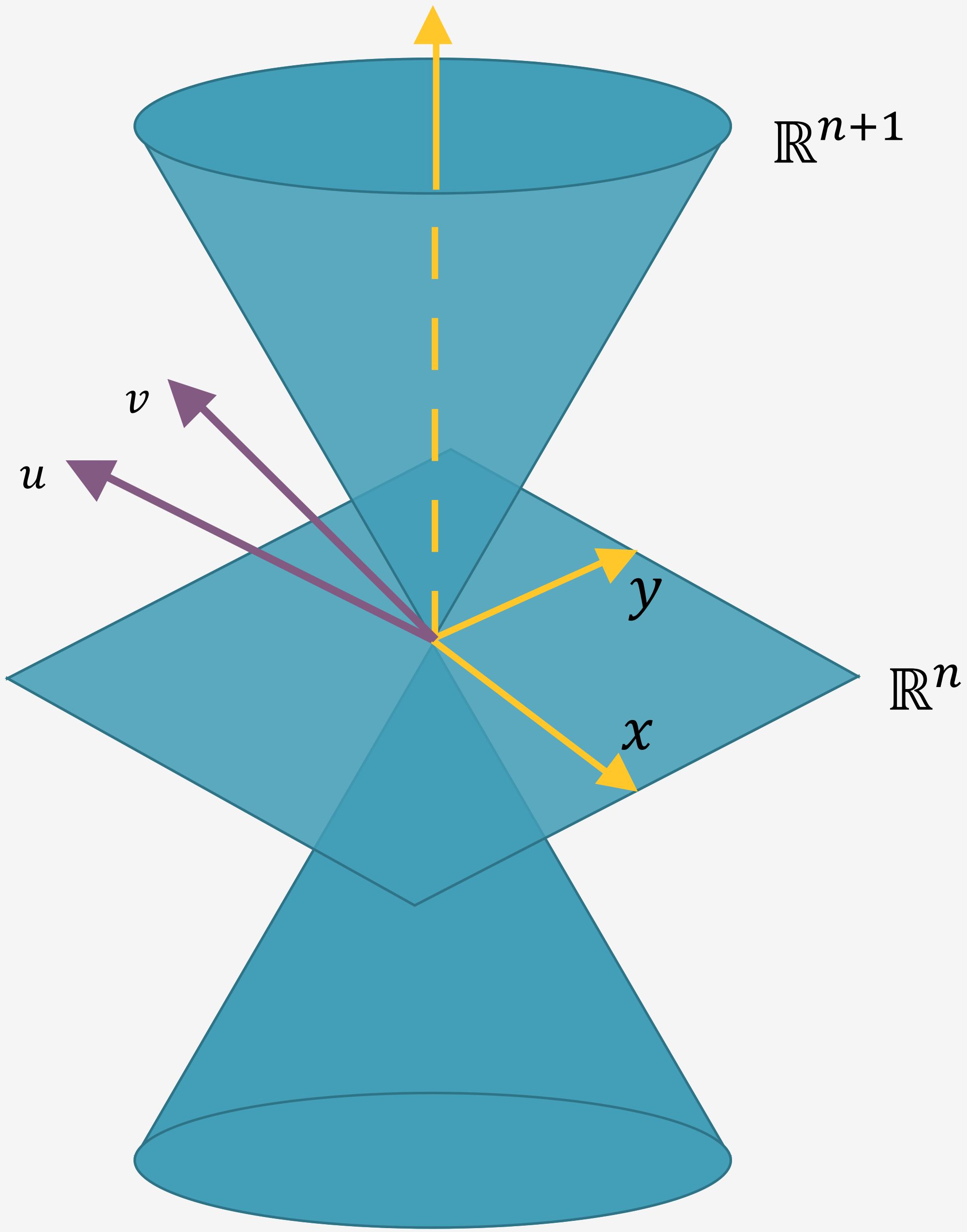
Lorentz Space



Lorentz Space

$\mathbb{R}^{n+1}, \langle \cdot \rangle$:

$$\langle u, v \rangle = u_1 v_1 + \cdots + u_n v_n - u_{n+1} v_{n+1}$$



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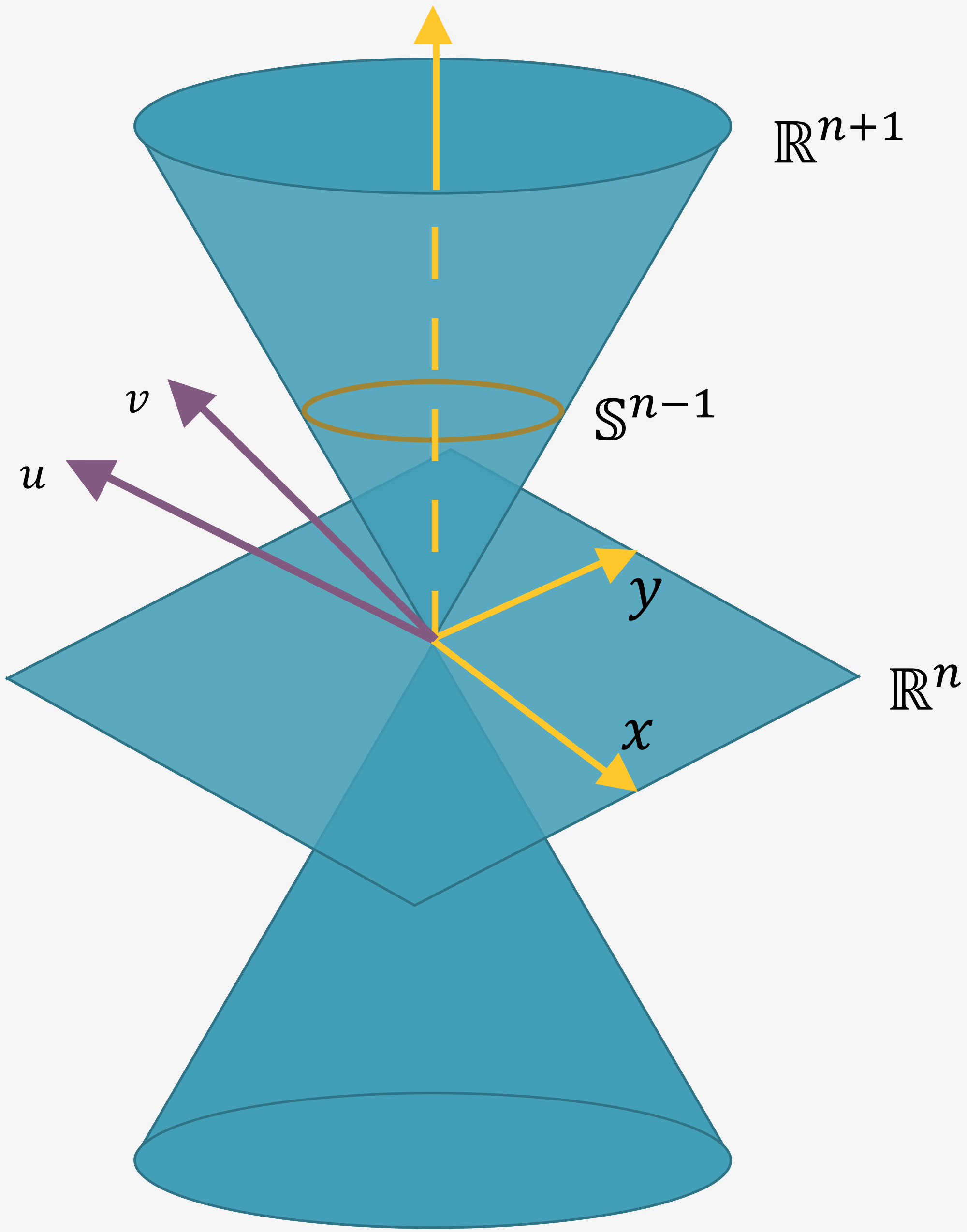

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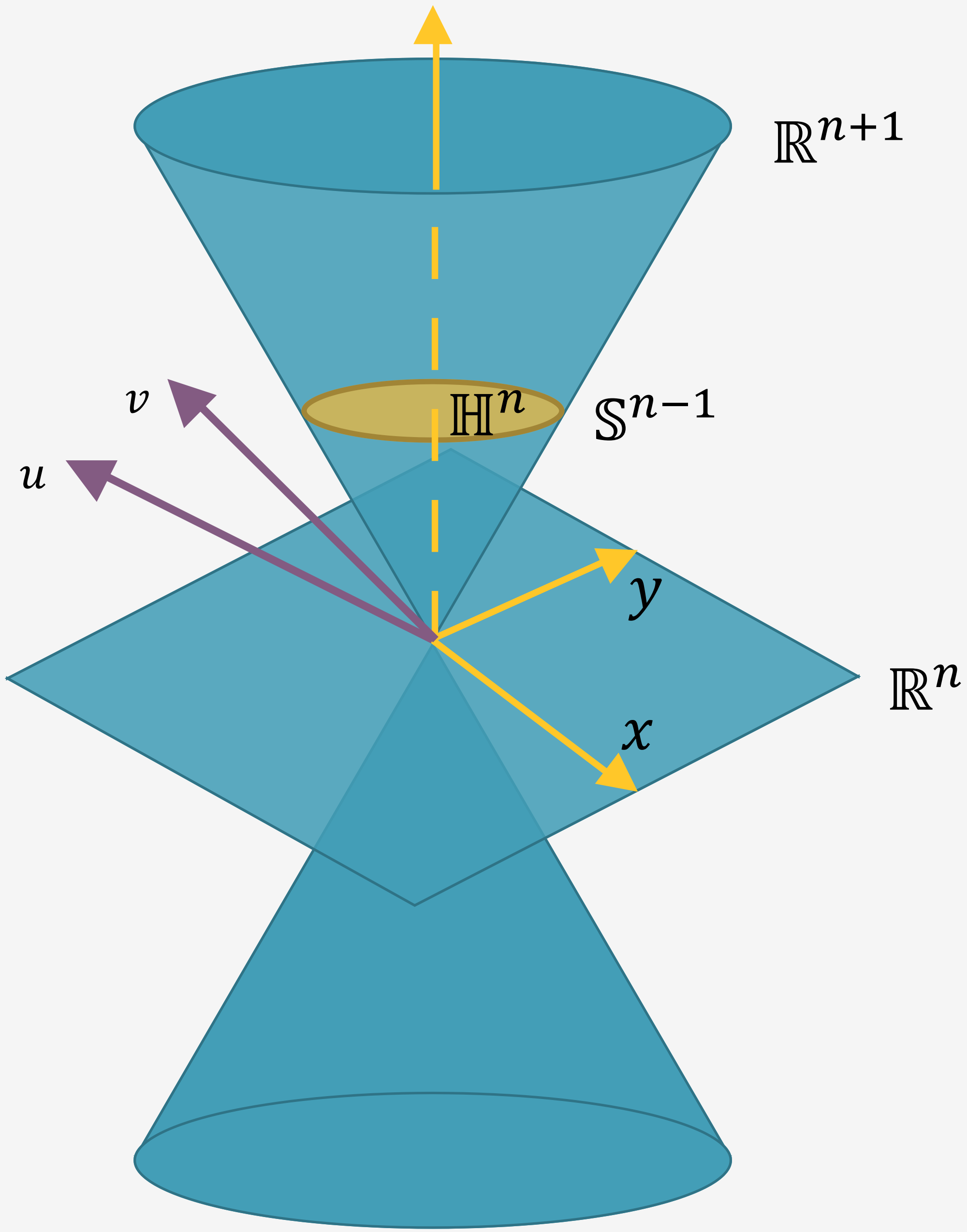

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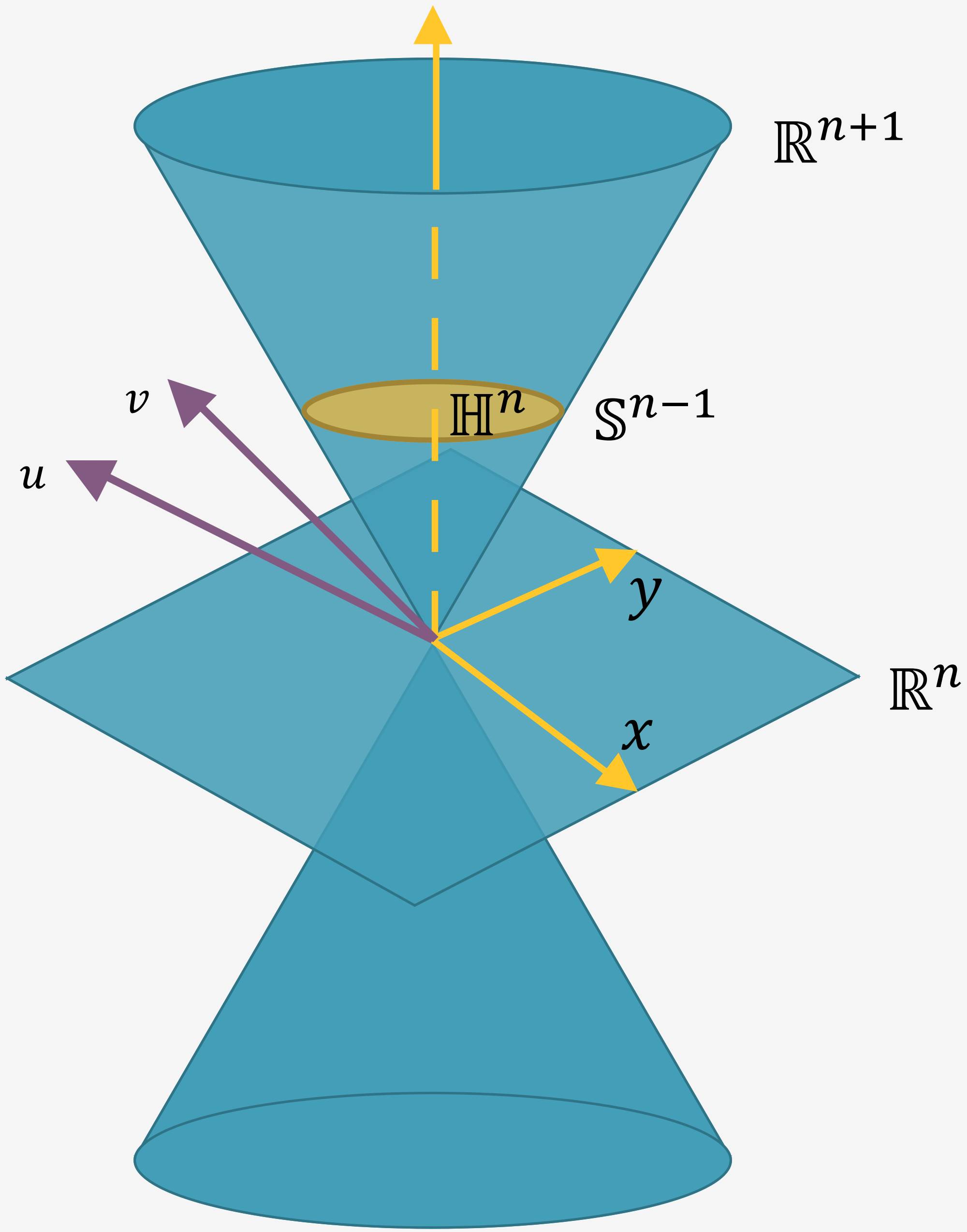
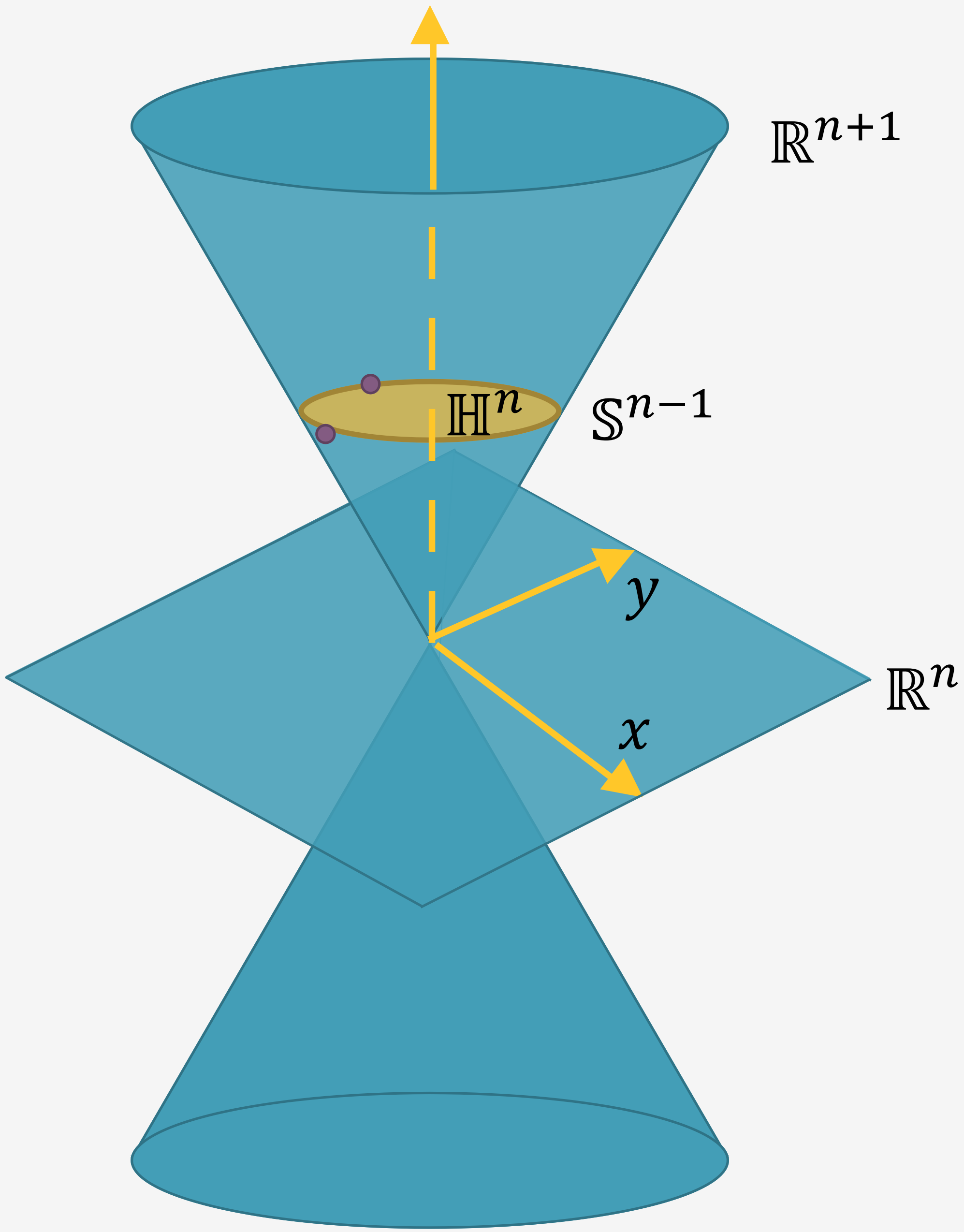
$$\mathrm{SO}^+(n,1) \cong \mathrm{Isom}^+(\mathbb{H}^n) \cong \mathrm{M\ddot{o}b}(\mathbb{S}^{n-1})$$



Lorentz Space

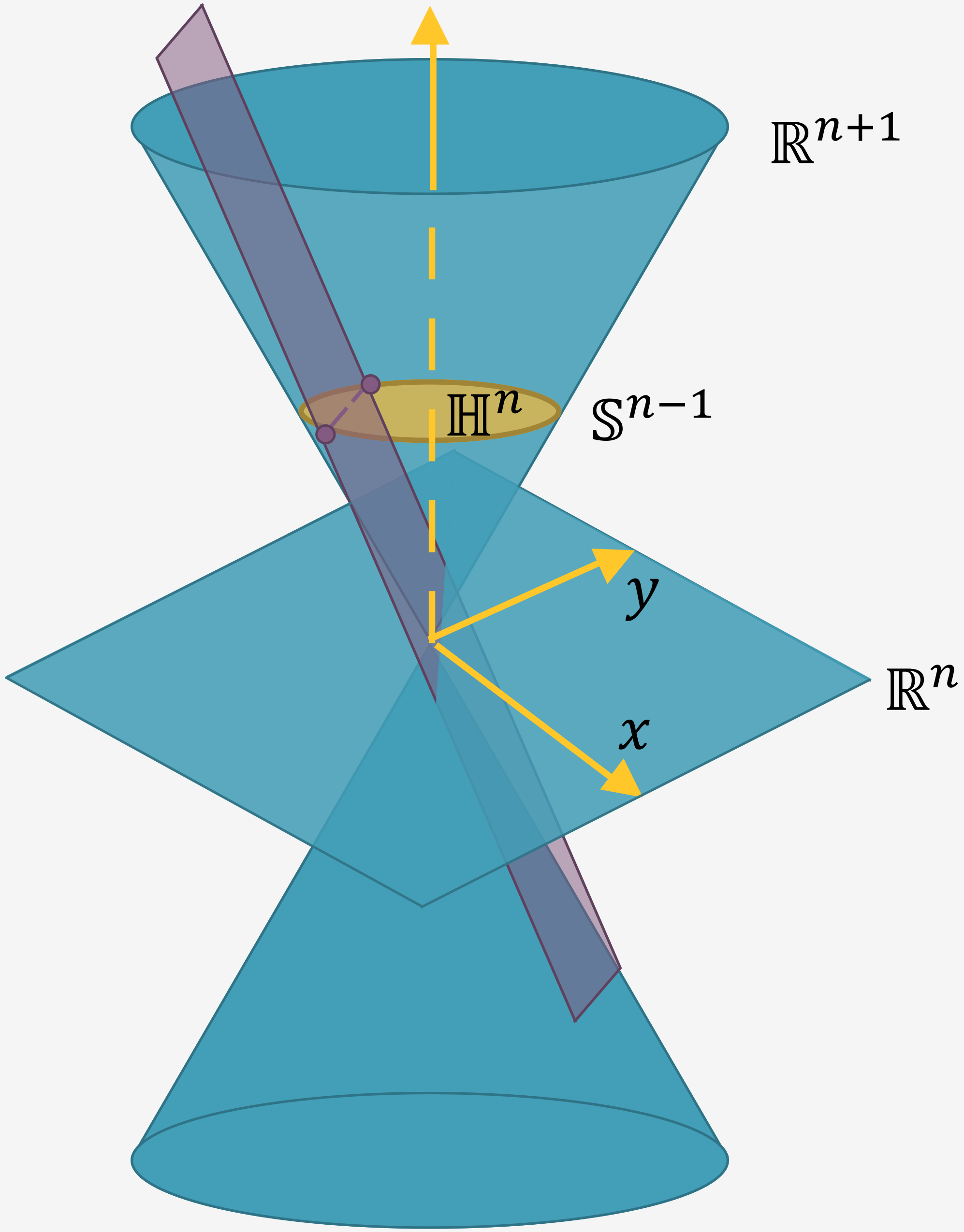
$(n-2)$-spheres in $\mathbb{S}^{n-1}$



Lorentz Space

$(n-2)$-spheres in $\mathbb{S}^{n-1}$ $\longleftrightarrow$

Time-like Lorentz subspaces of $\mathbb{R}^{n+1}$ $\longleftrightarrow$



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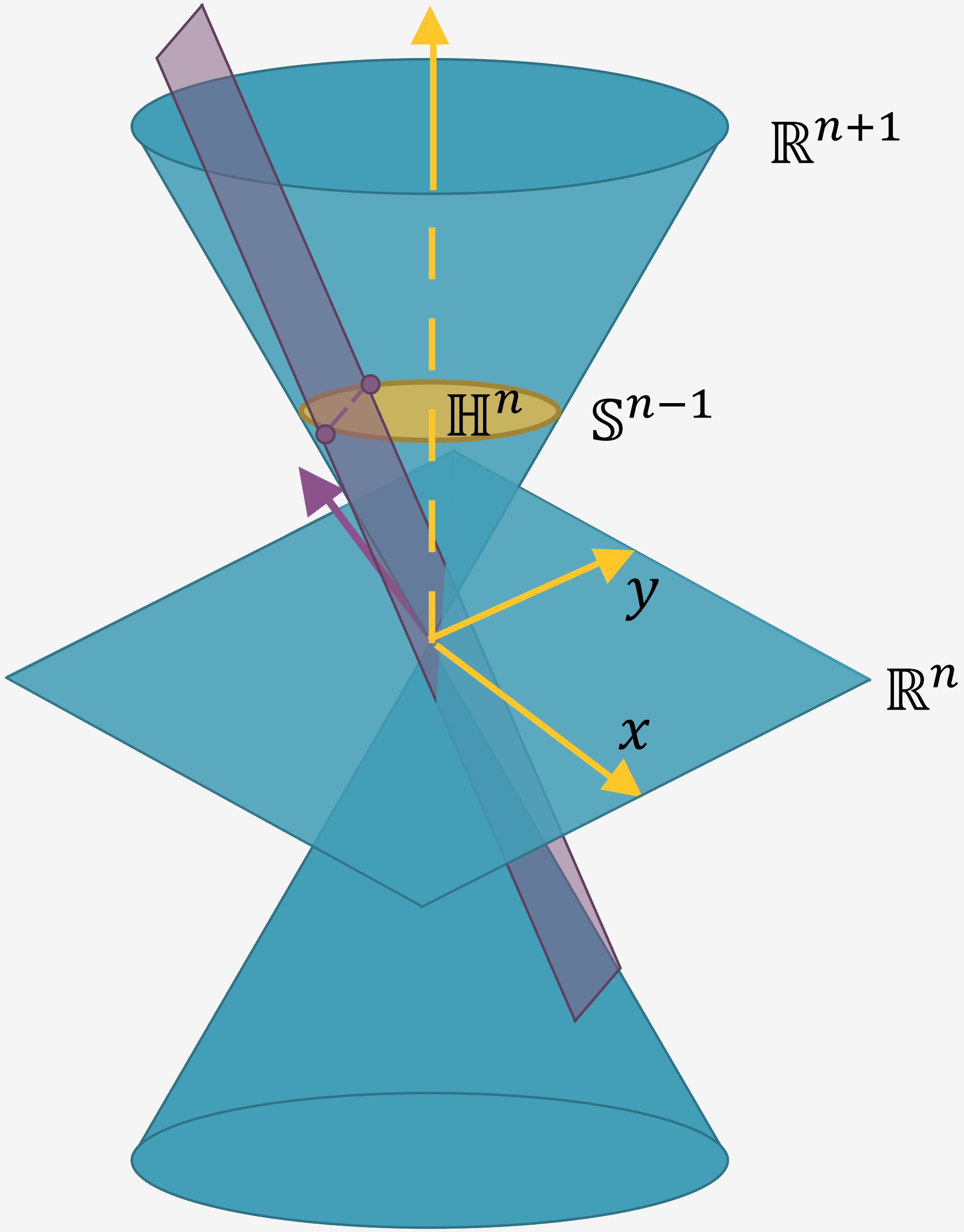

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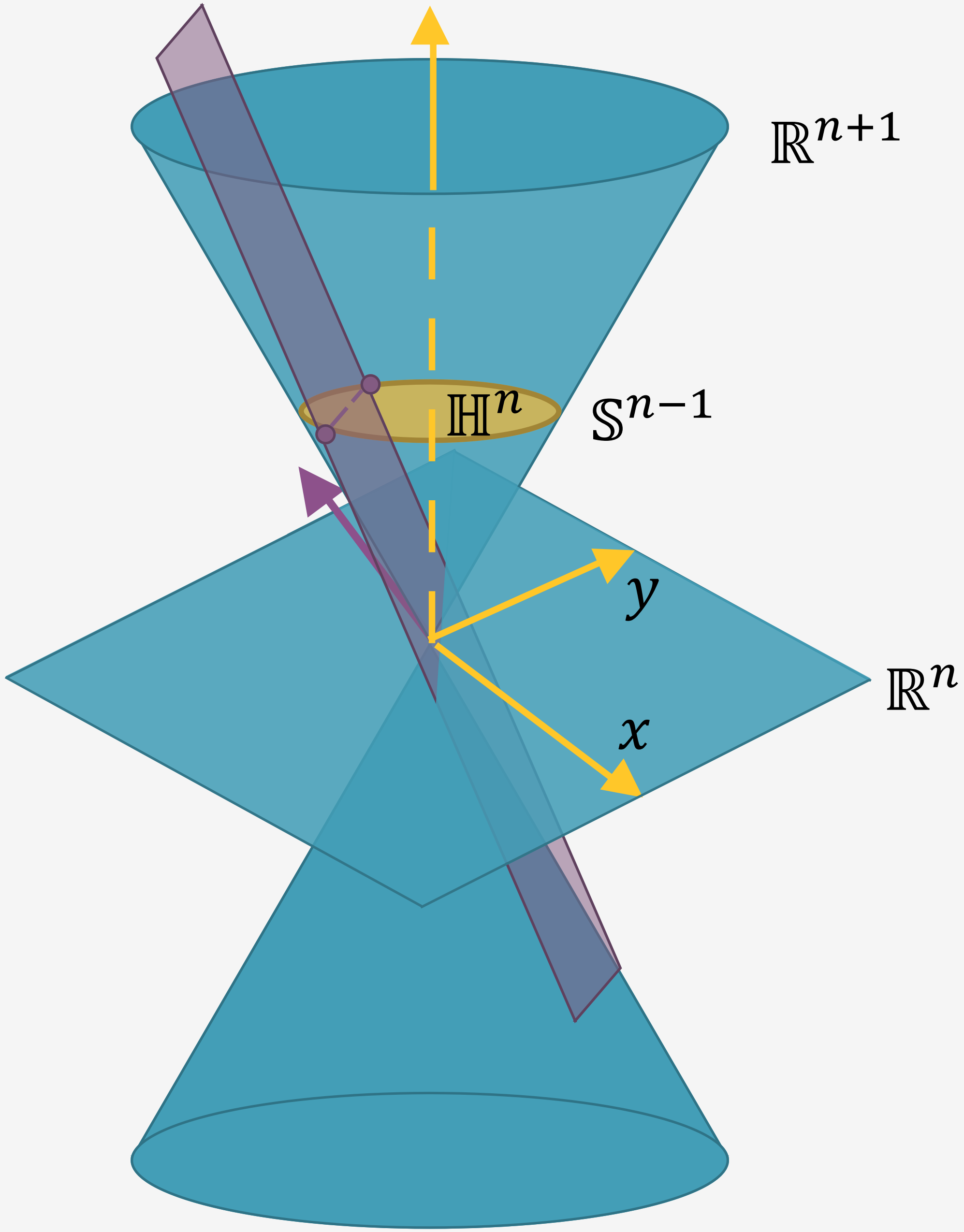
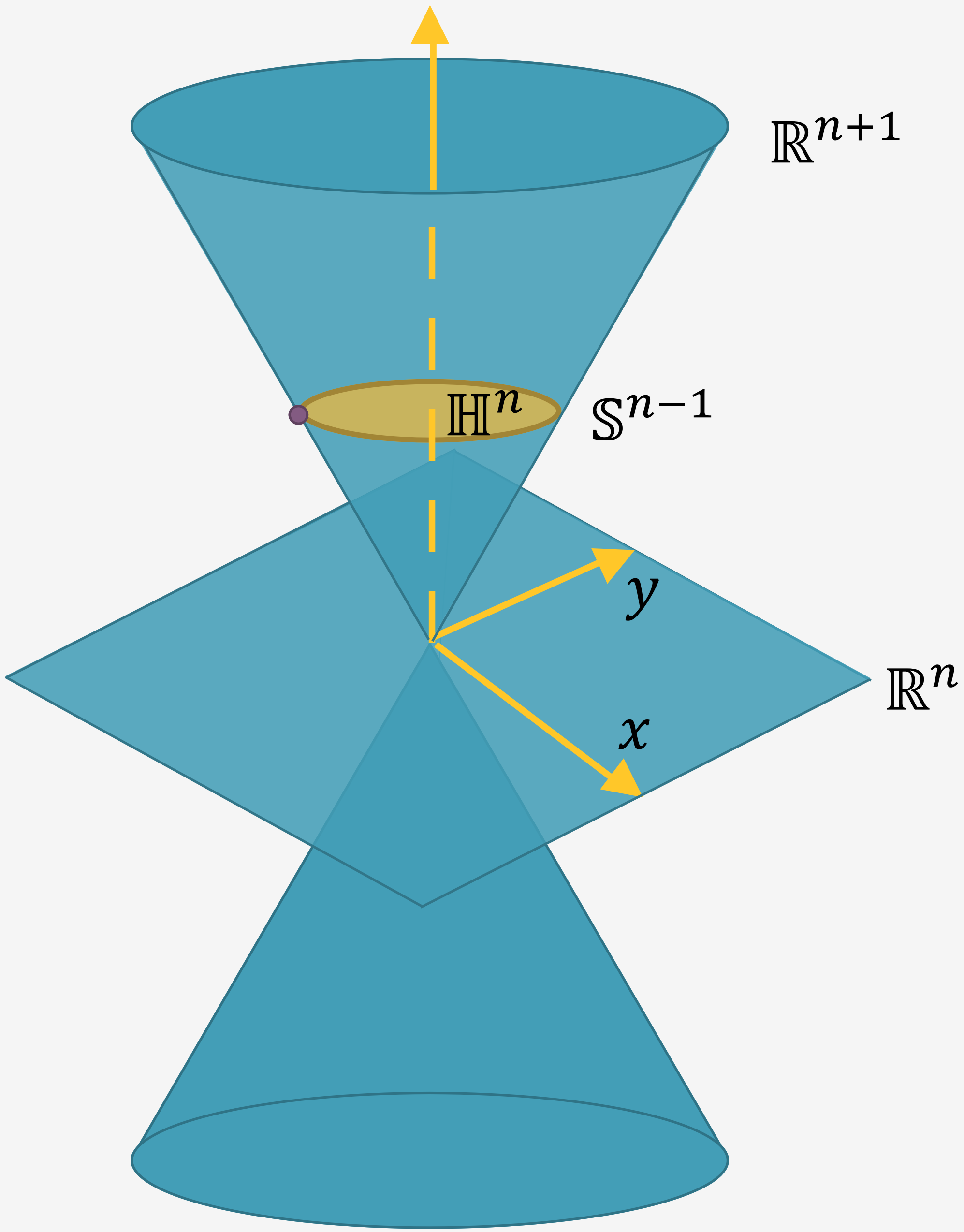


Fact: Lorentz inner product of two space-like vectors is equal to inversive distance of the corresponding $(n-2)$-spheres.

Lorentz Space

Ideal points in $\mathbb{S}^{n-1}$



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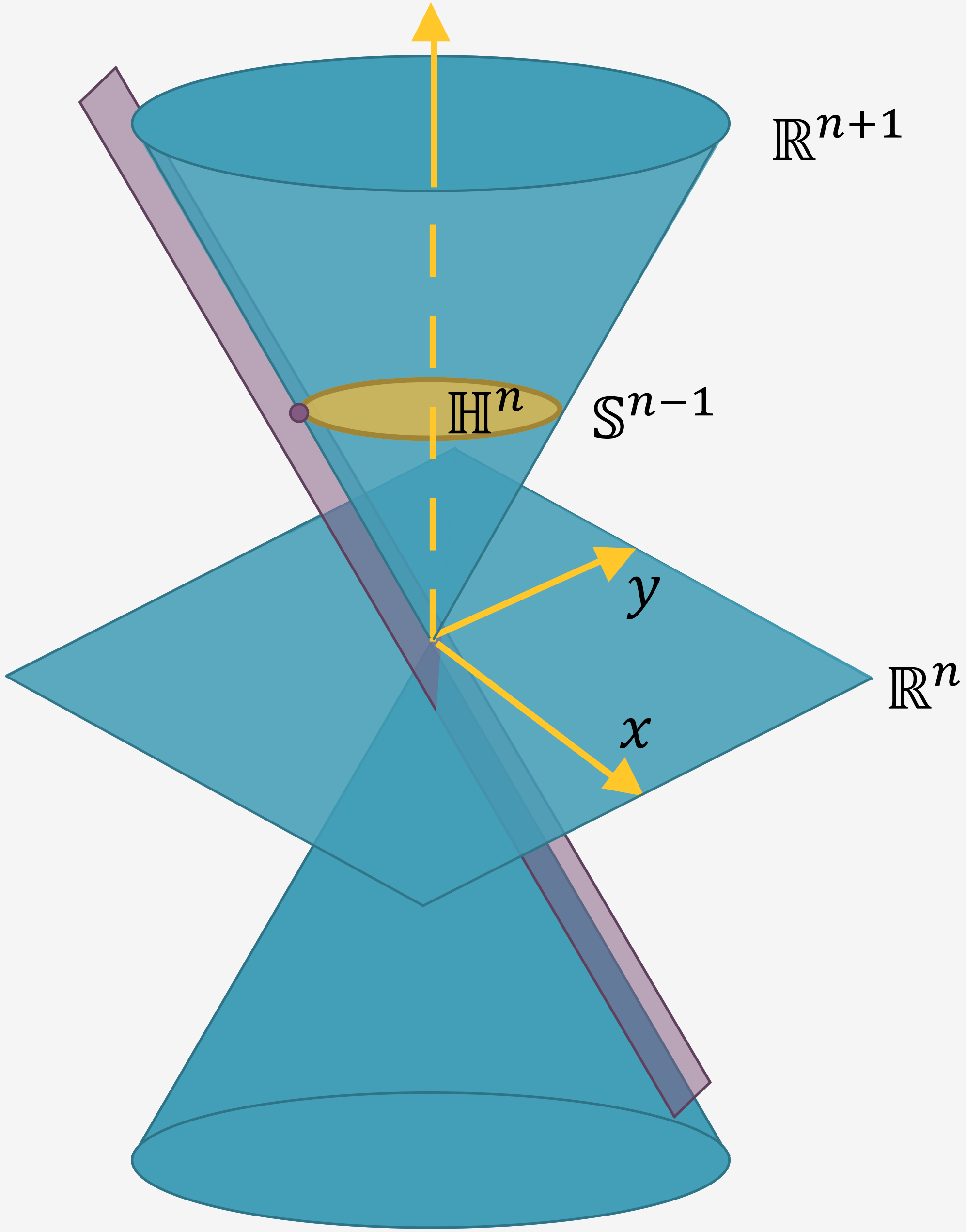
Ideal points in $\mathbb{S}^{n-1}$ $\longleftrightarrow$

Light-like Lorentz Subspaces



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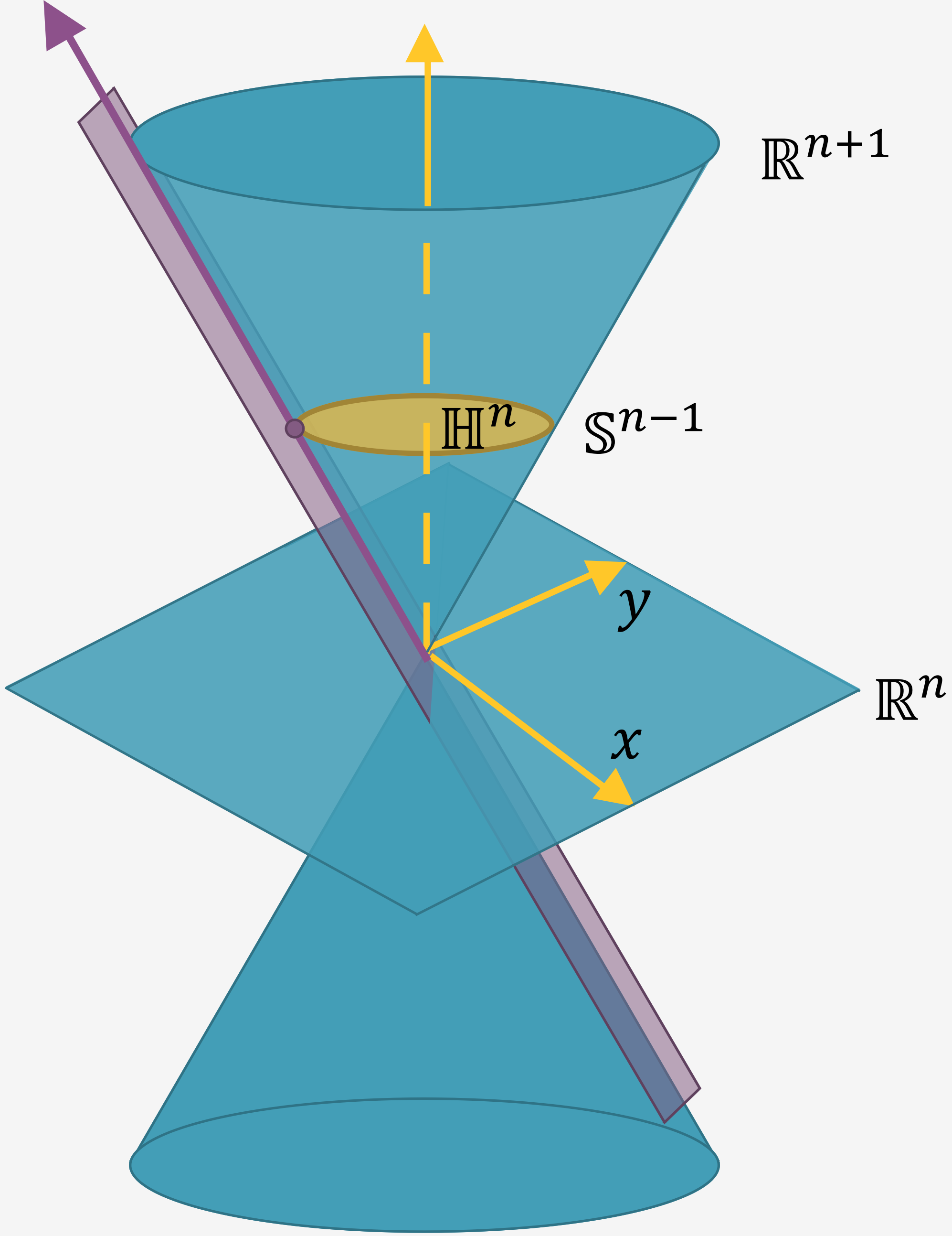

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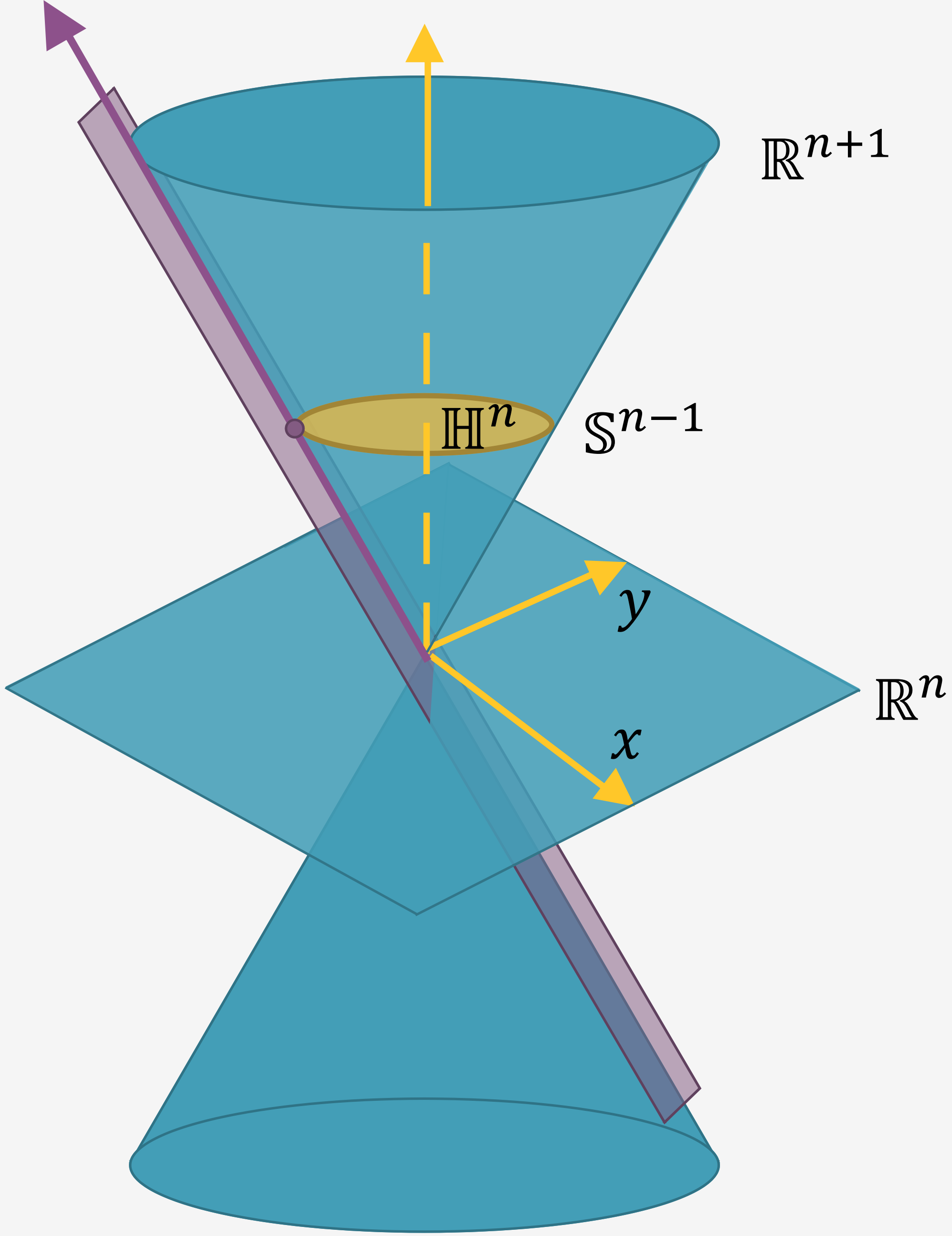
Lorentz Inner Product:
Absolute Cross Ratio



Linearly Independent Lorentz Vectors

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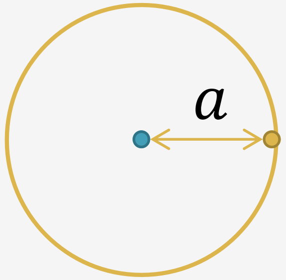

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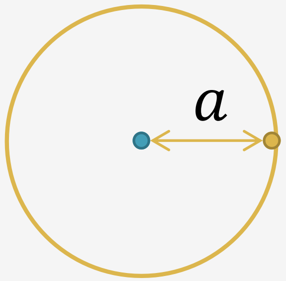

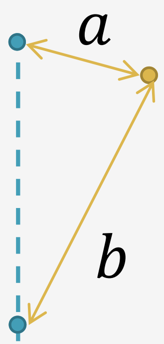

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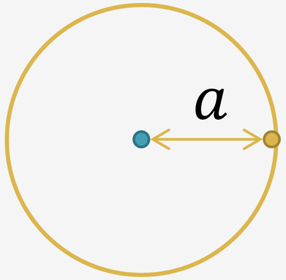

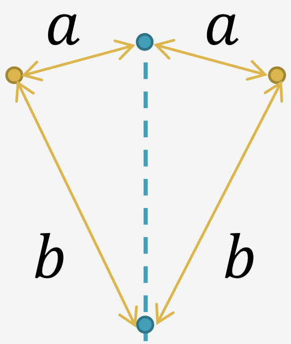

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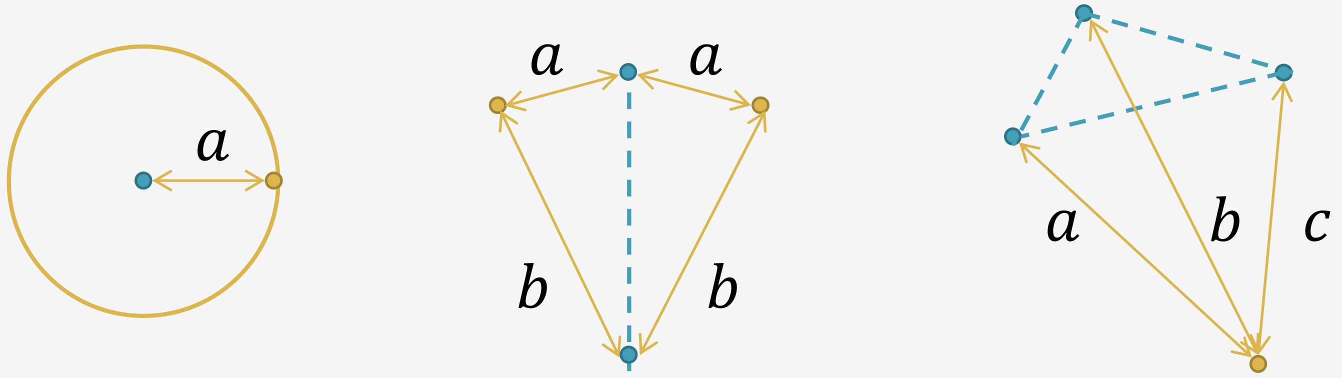
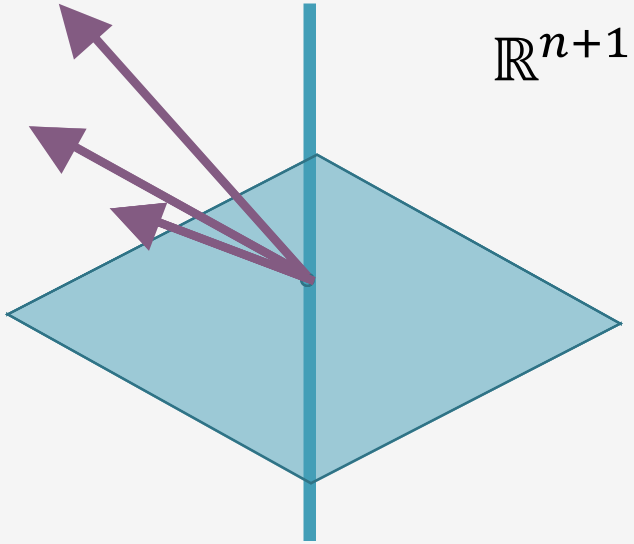
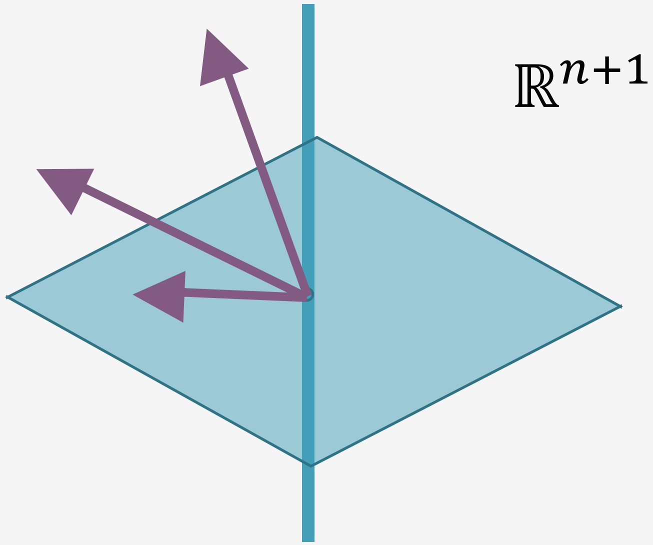
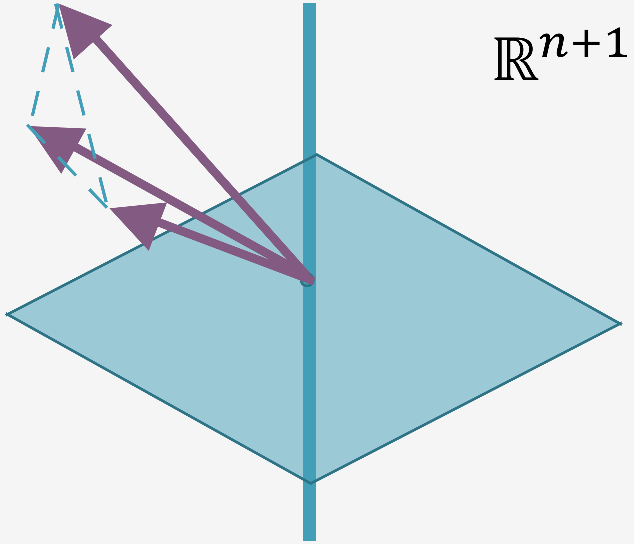
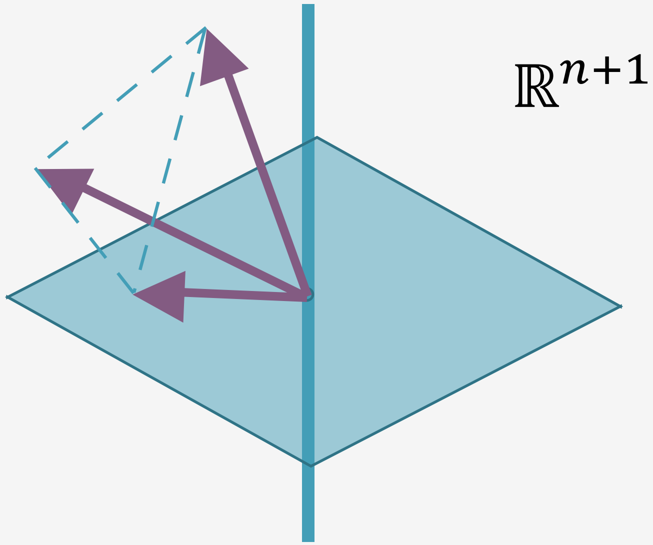




New Rigidity Statement







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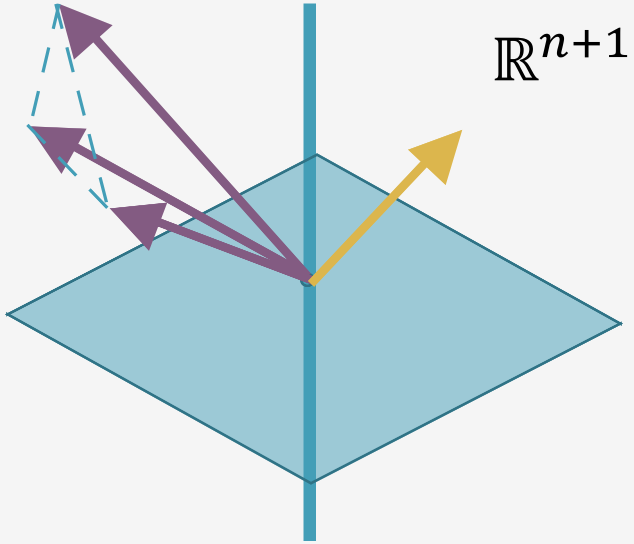

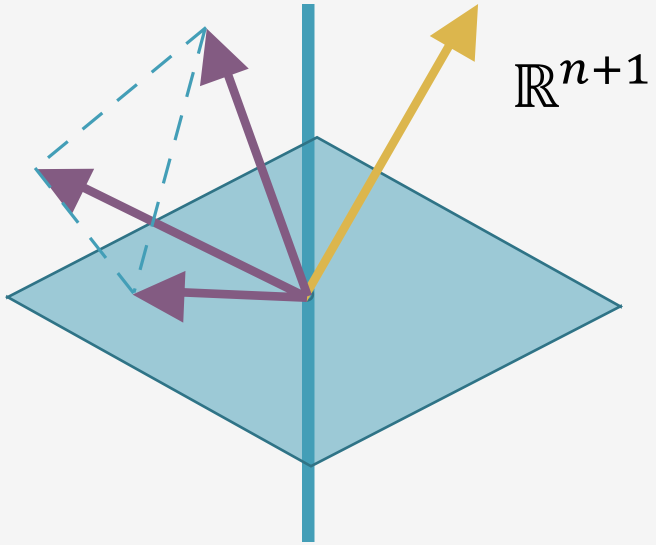

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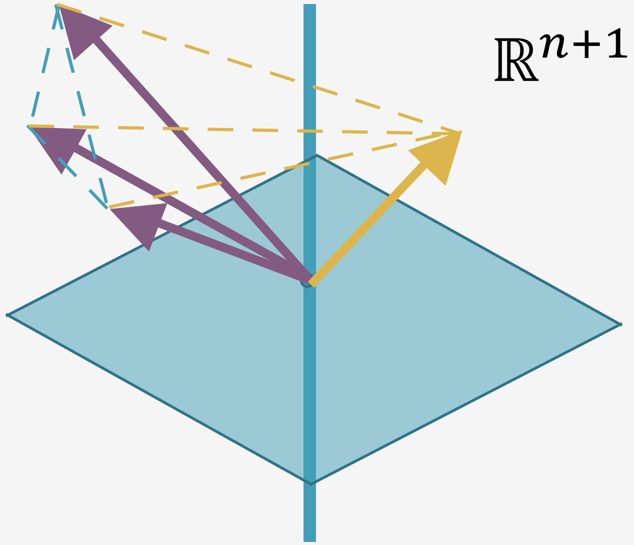

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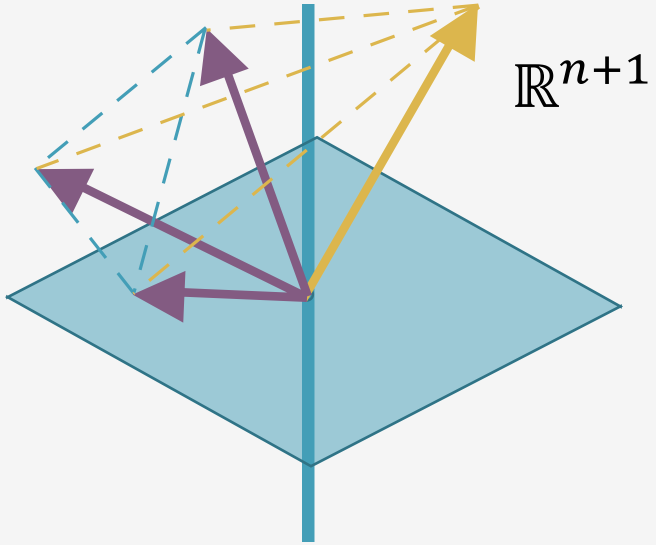

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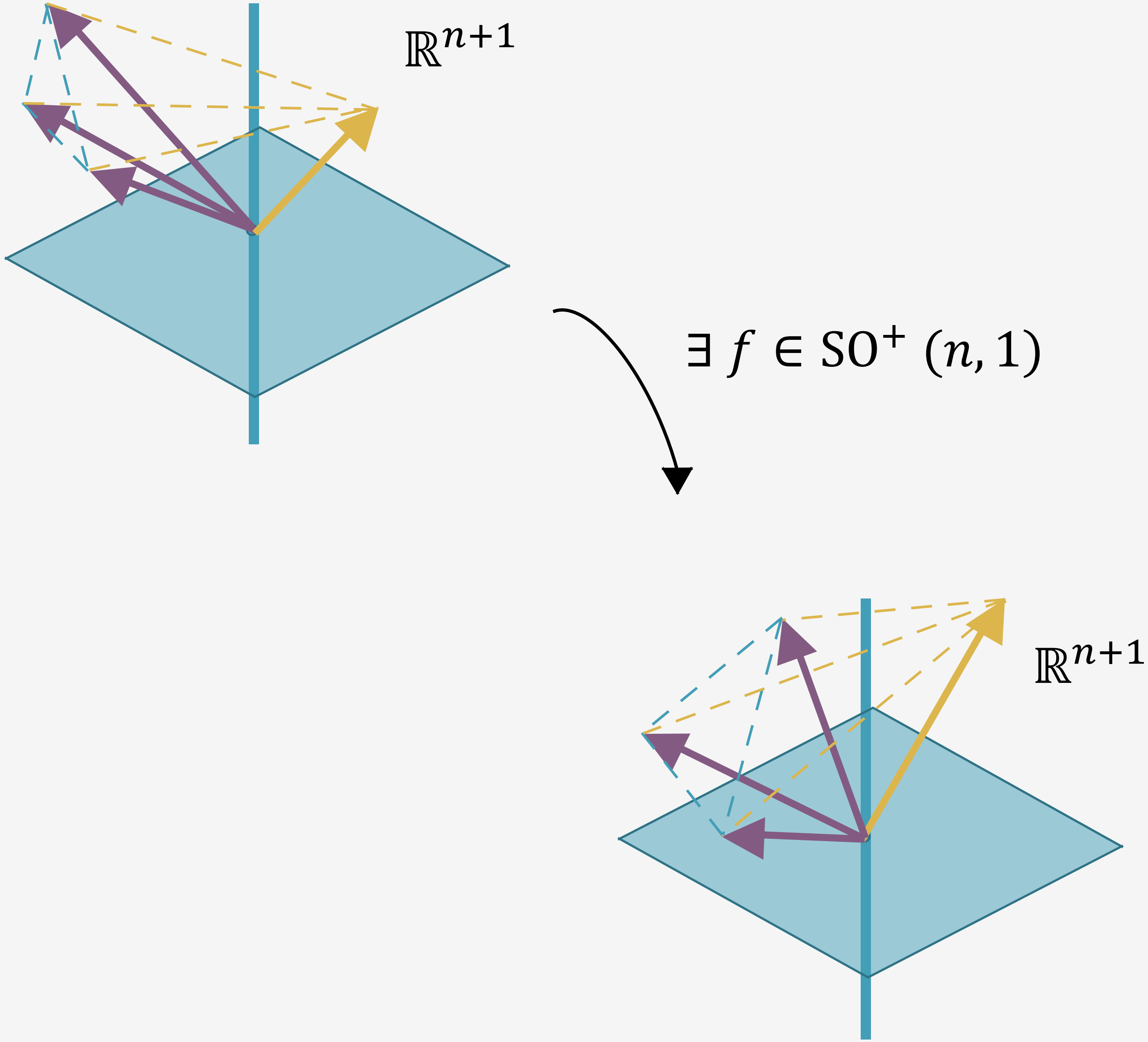
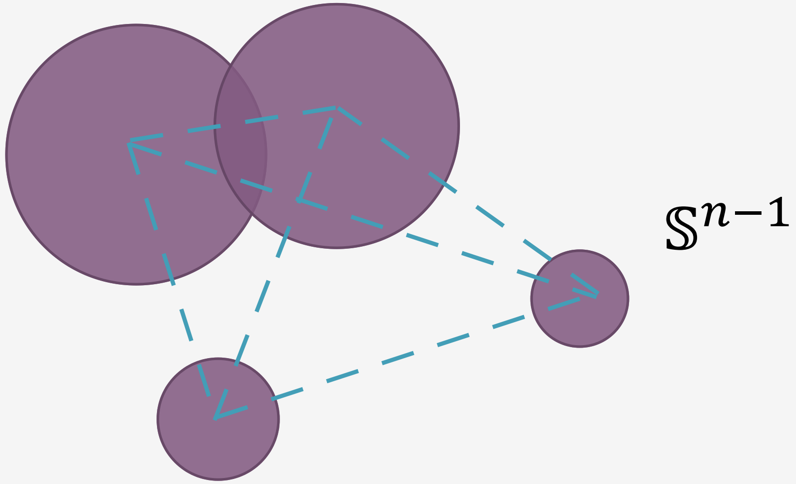
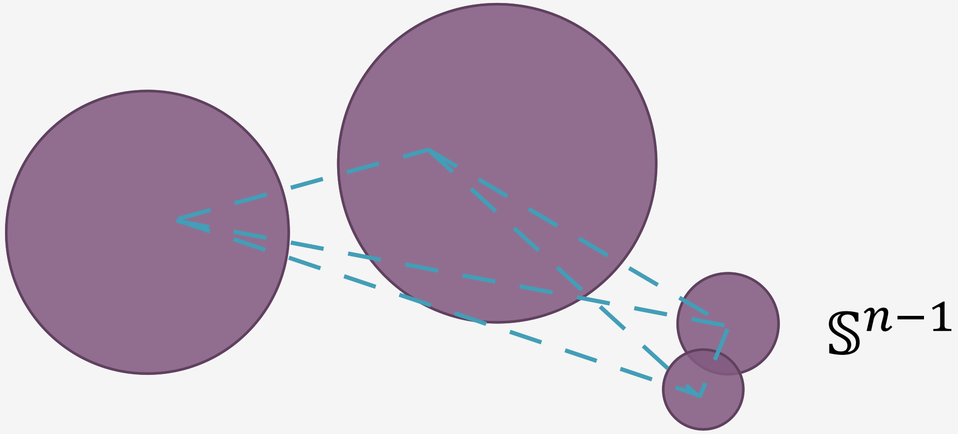


New Rigidity Statement – Spheres





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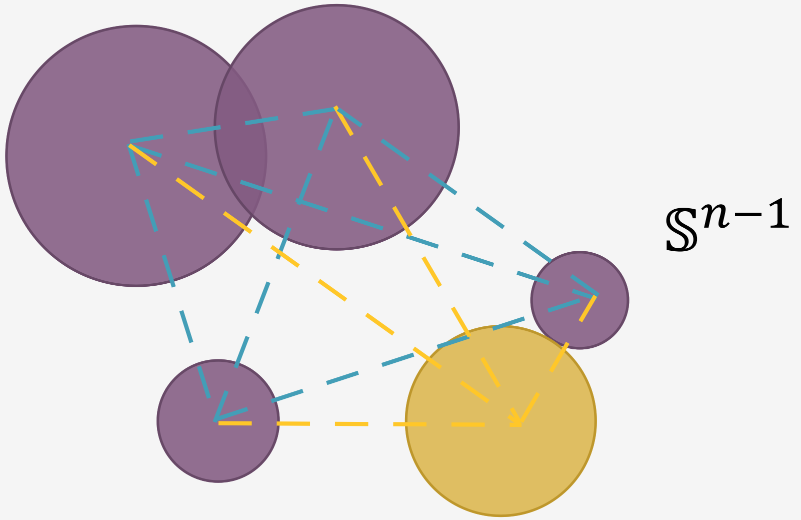

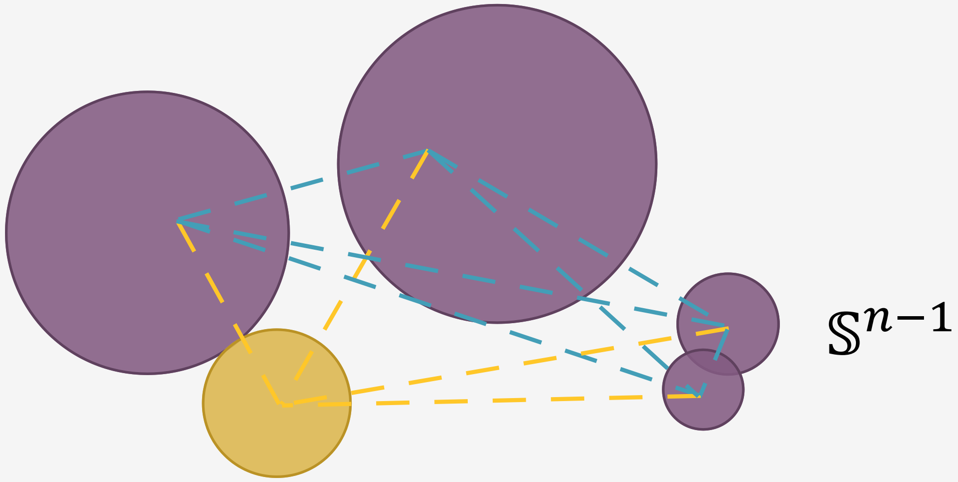

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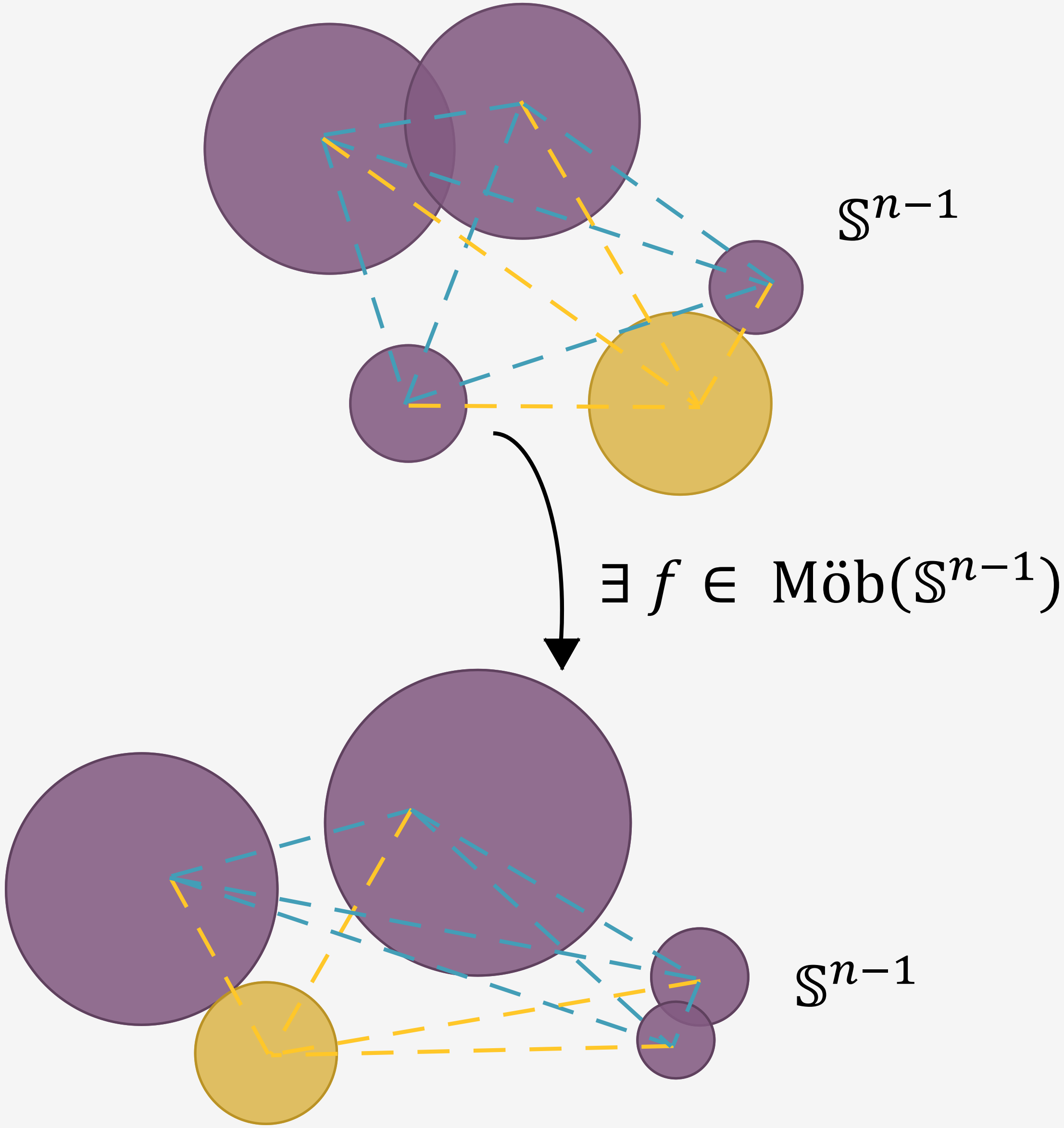

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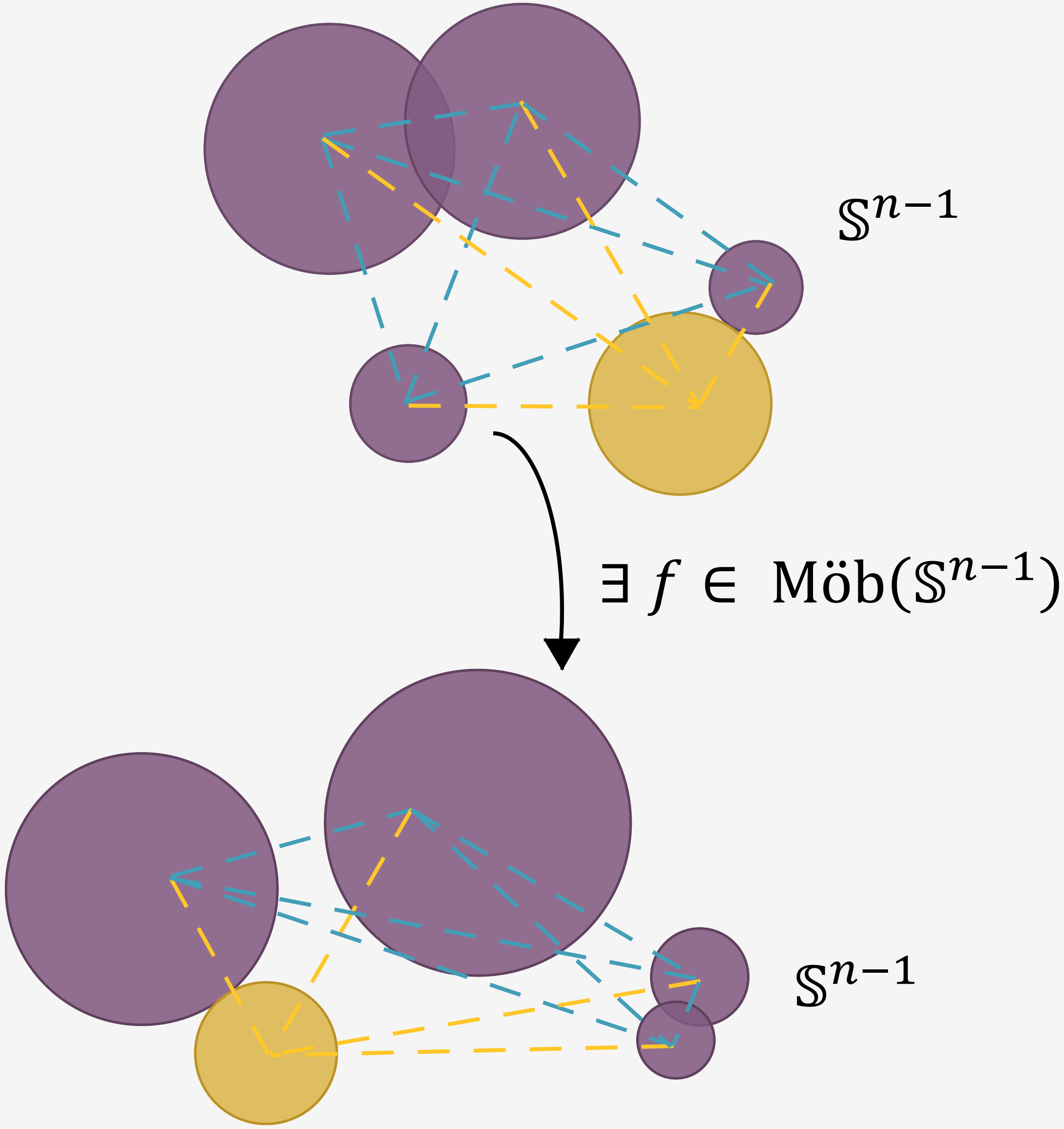

New Rigidity Statement—Points

- **Ideal Points:** Only need $|p_1, p_2, p_3, p_\alpha| = |p_1', p_2', p_3', p_\alpha'|$ for chosen p_1, p_2, p_3 in independent subcollection.

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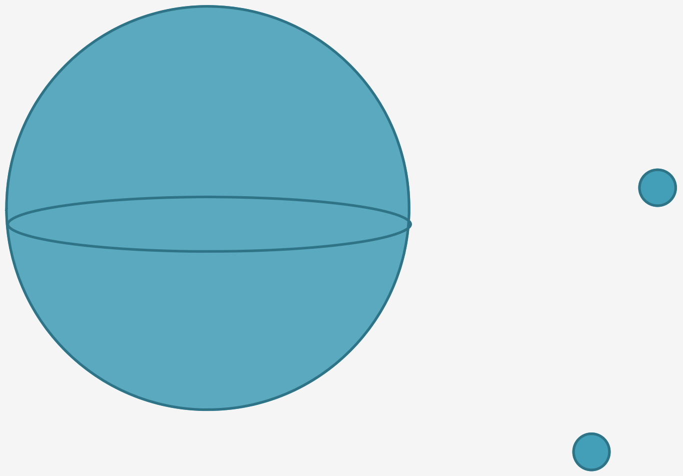

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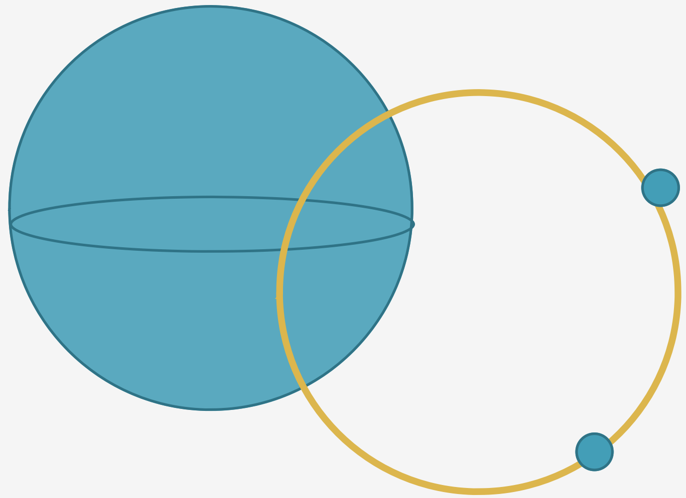

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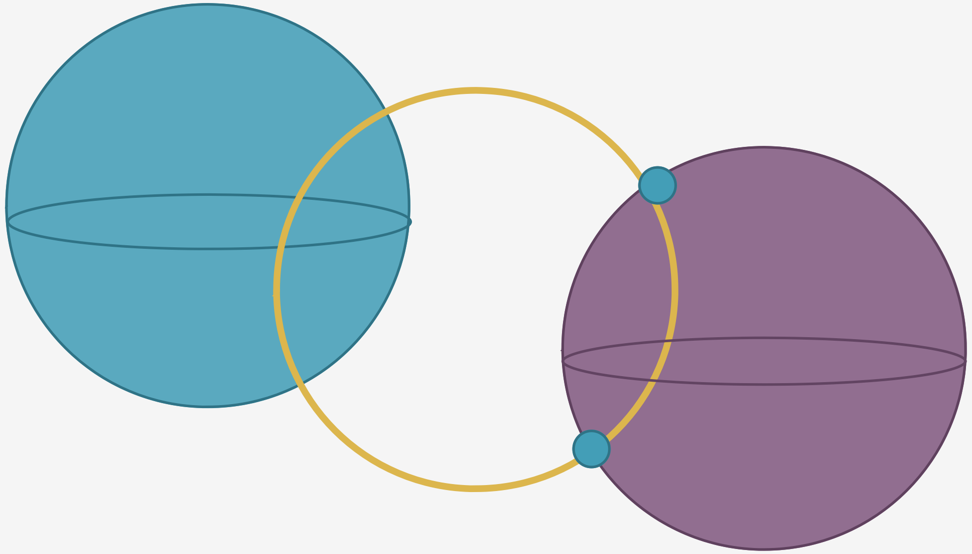
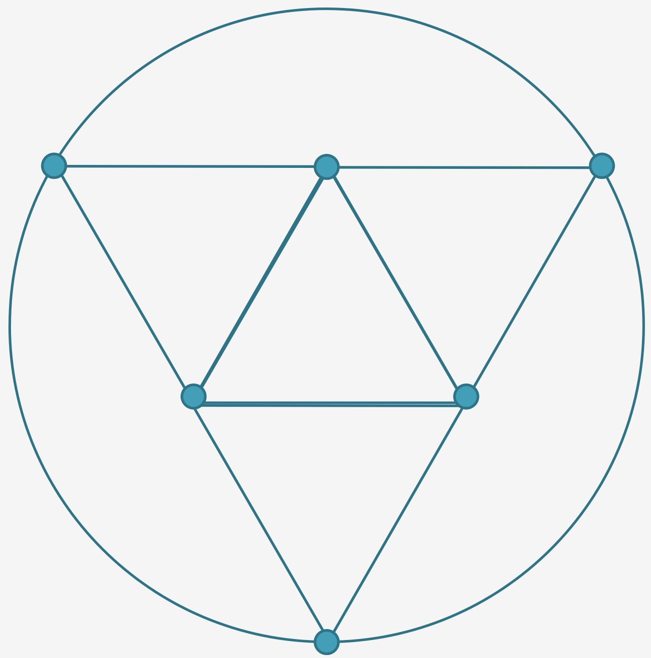


Inversive Distance Circle Packings



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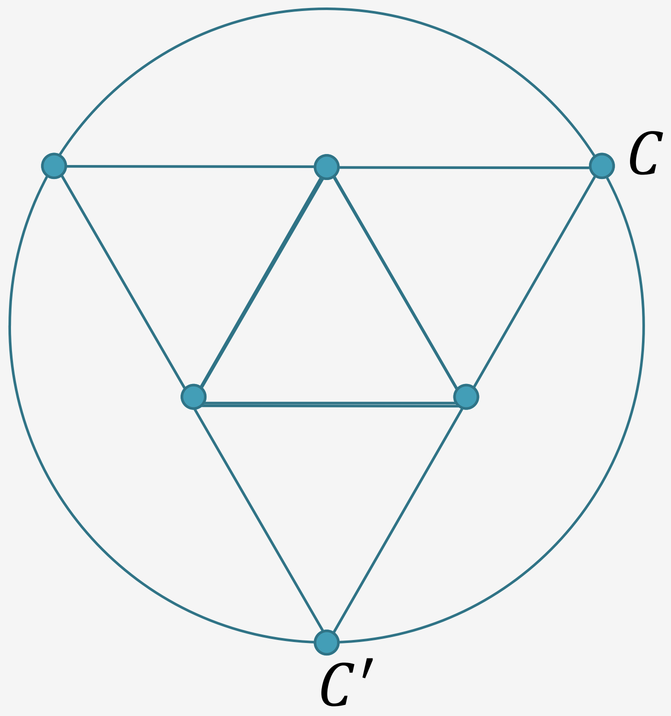

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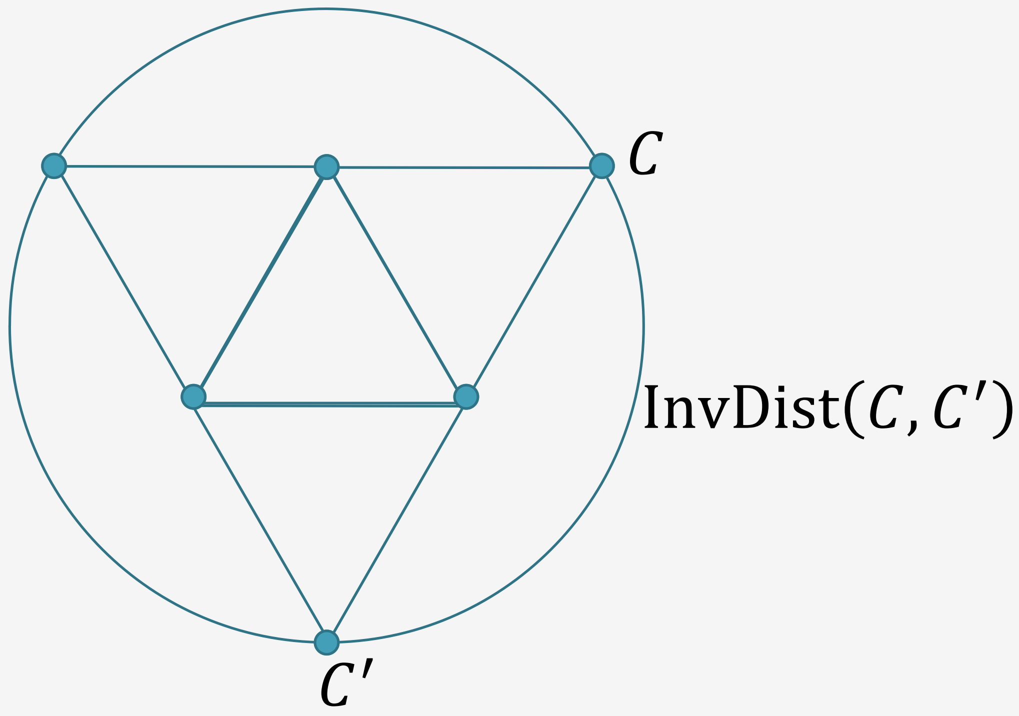
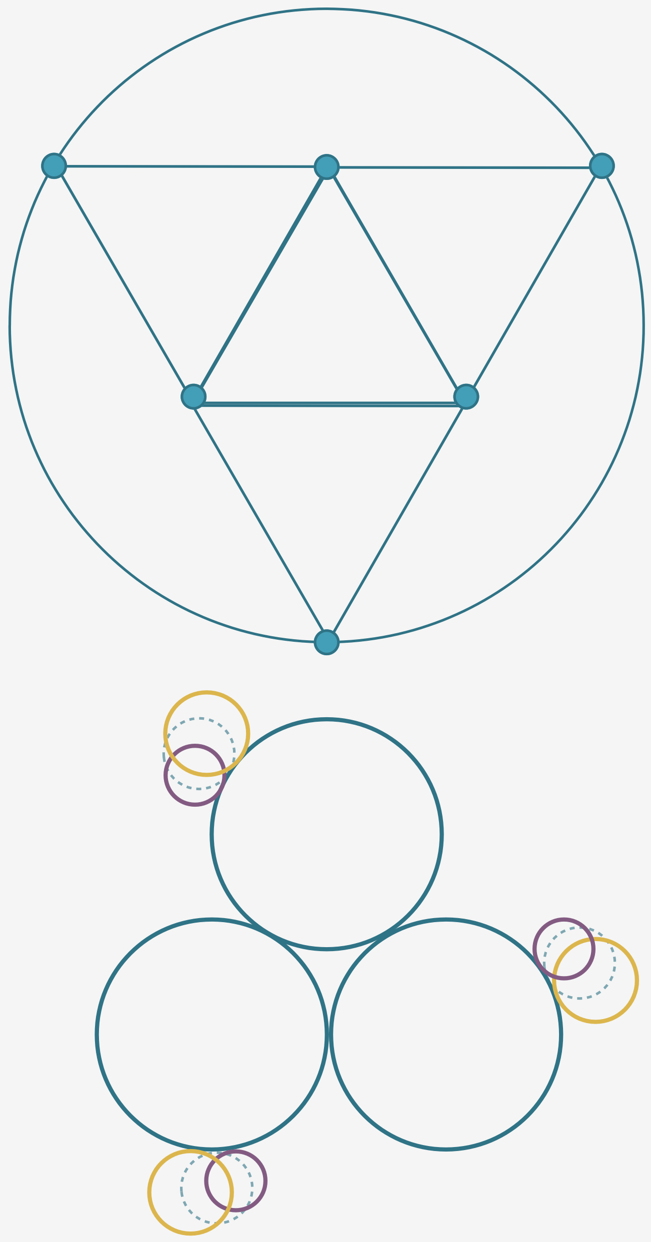


Inversive Distance Circle Packings

Ma-Schlenker C-Octahedra in the 2-Sphere
(Bowers & Bowers 2018)



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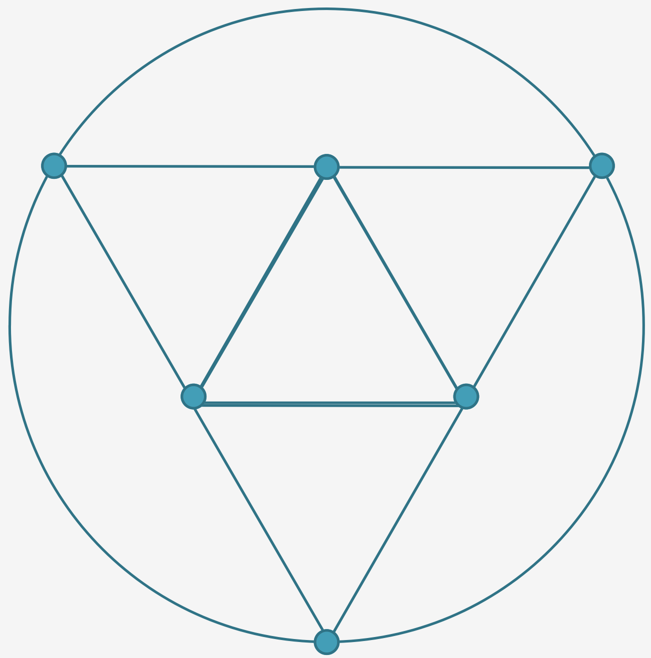

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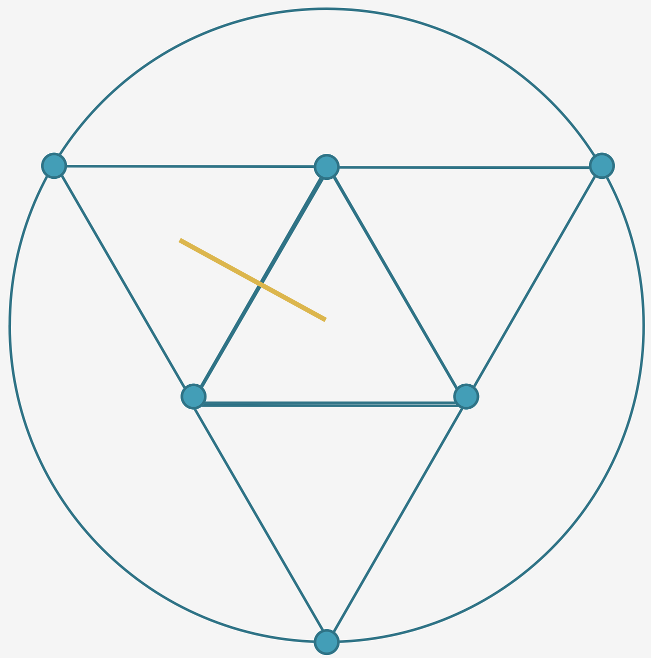

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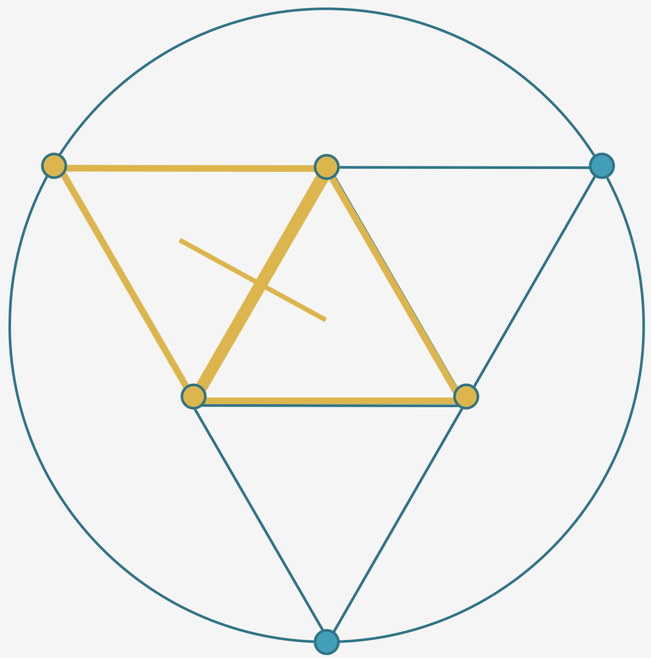

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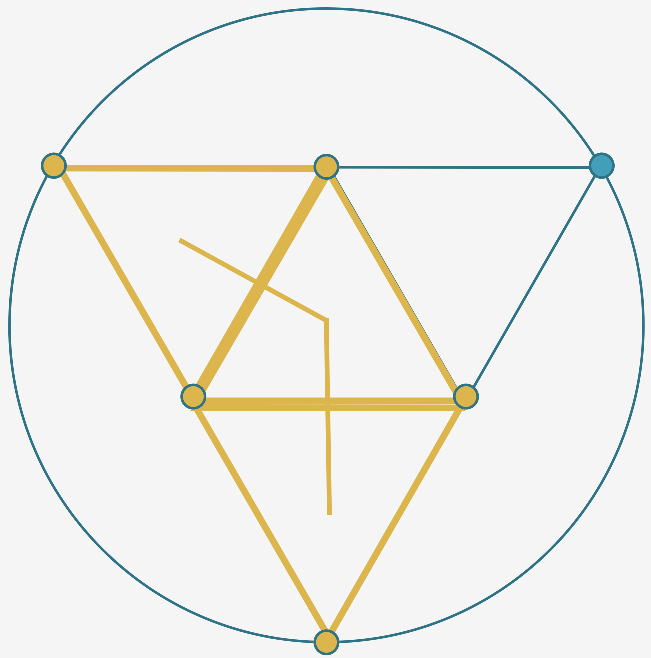

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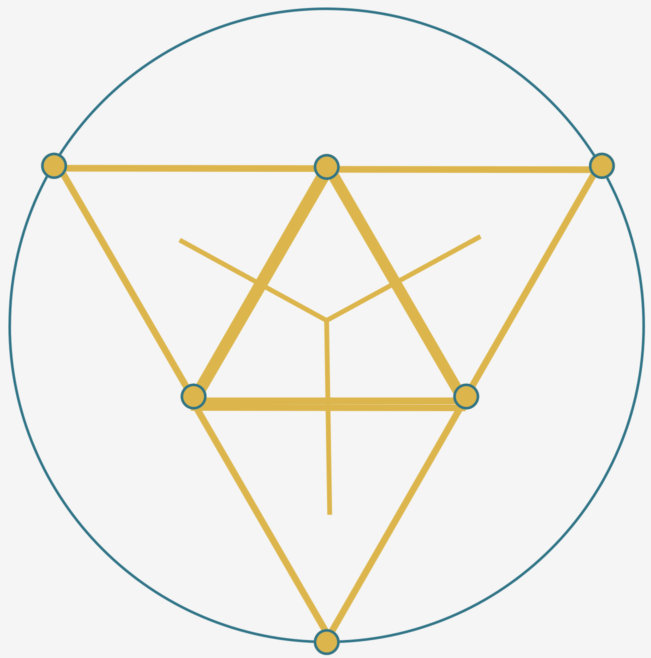

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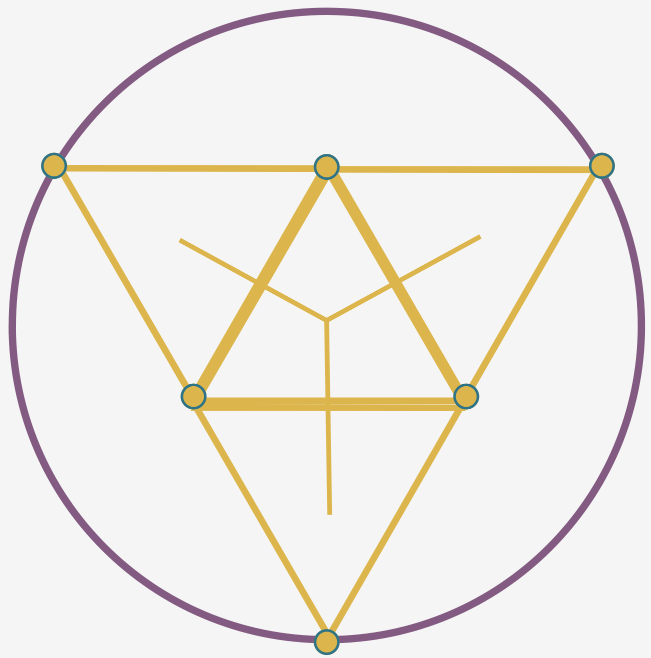
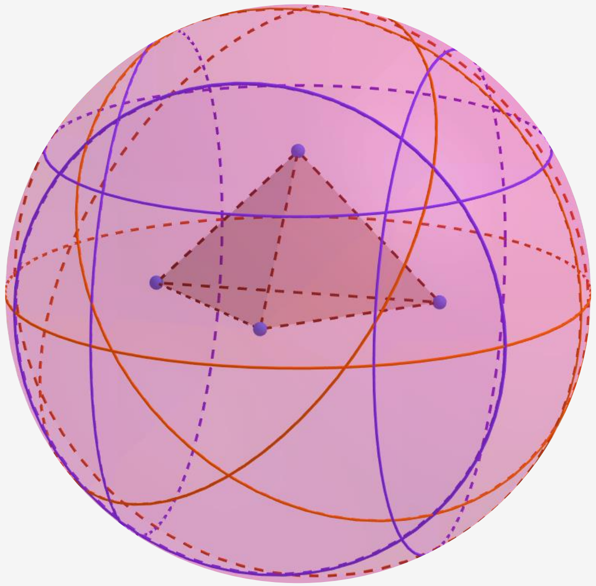
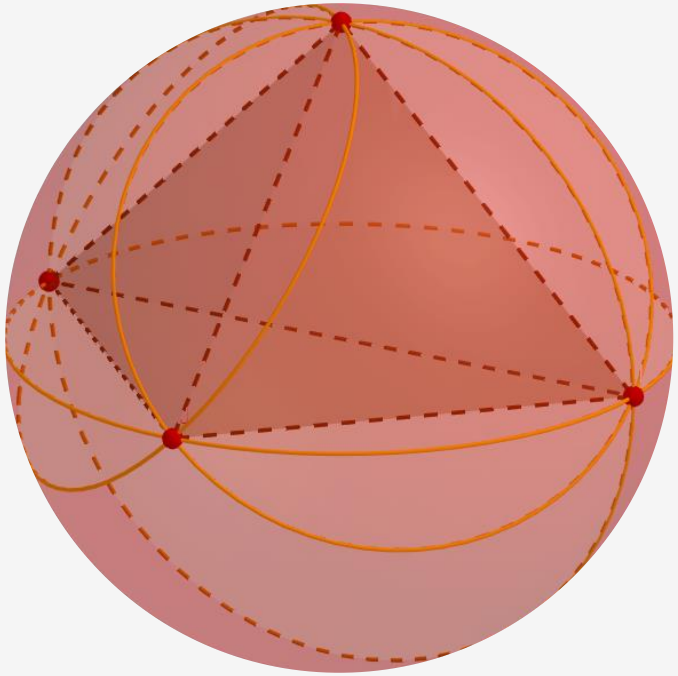
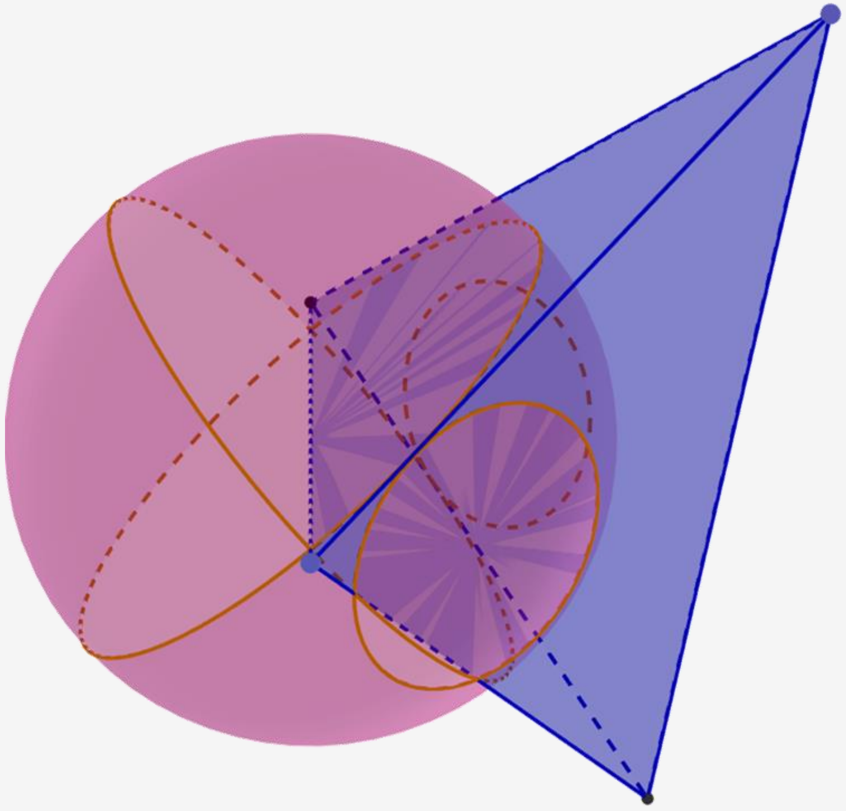


Projective Polyhedra

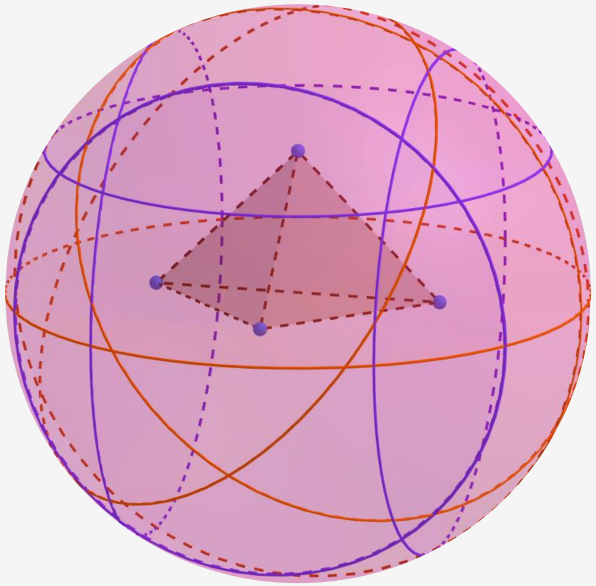


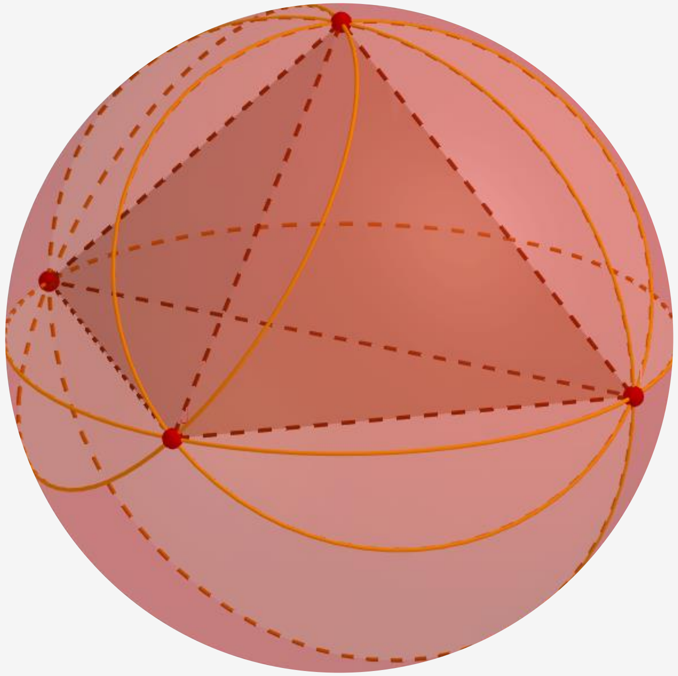


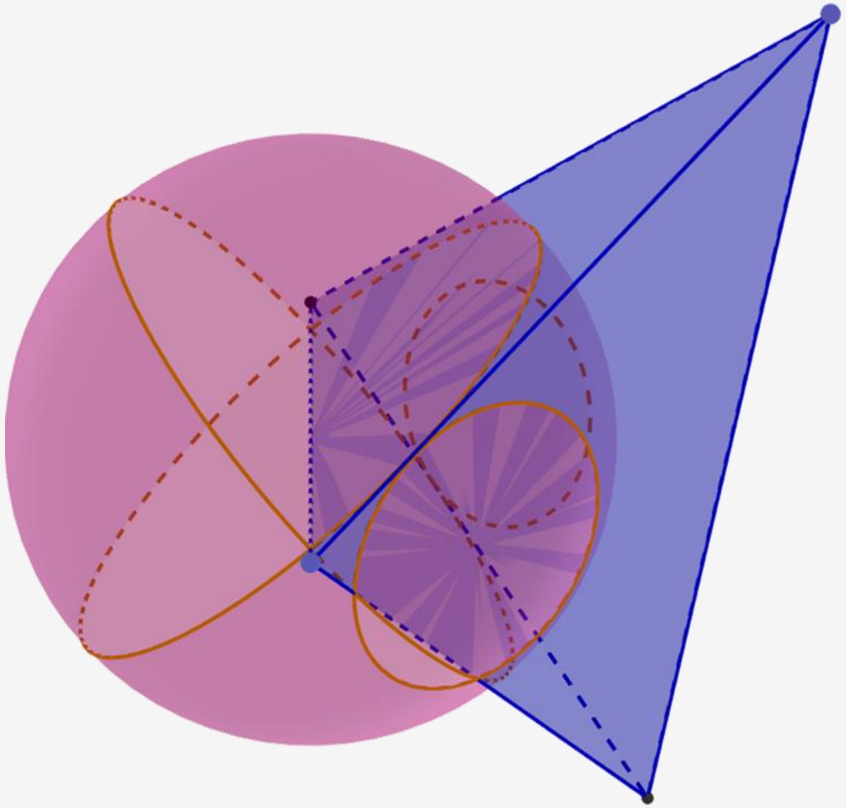


Projective Polyhedra

Andre'ev ('70);
Rivin & Hodgson ('93)







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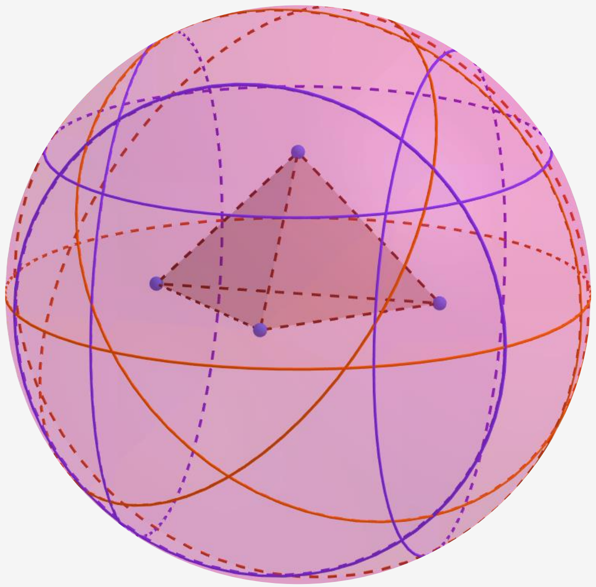
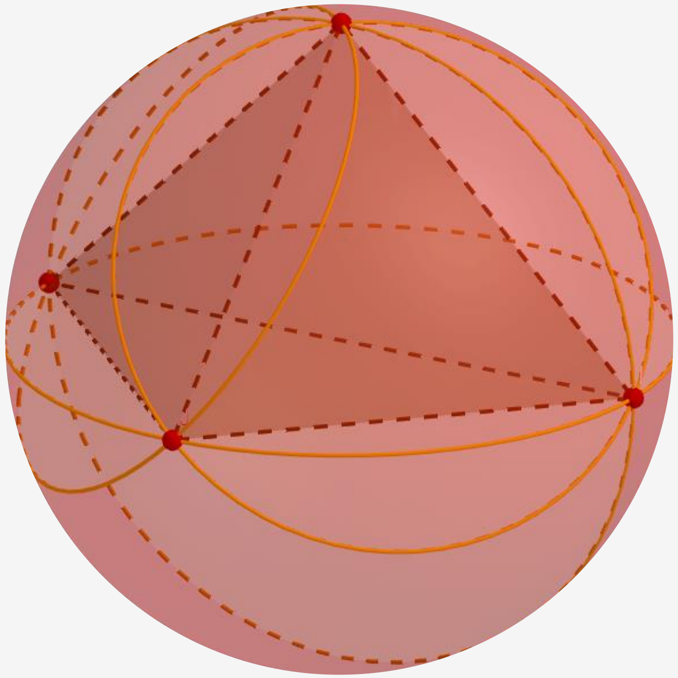
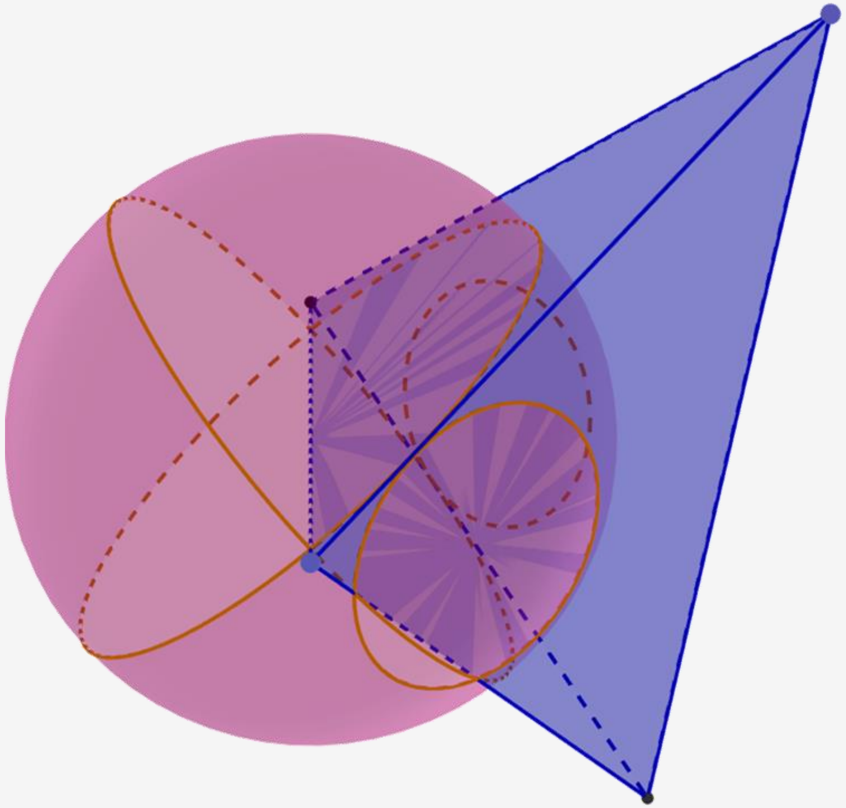


Rivin ('96)





Projective Polyhedra

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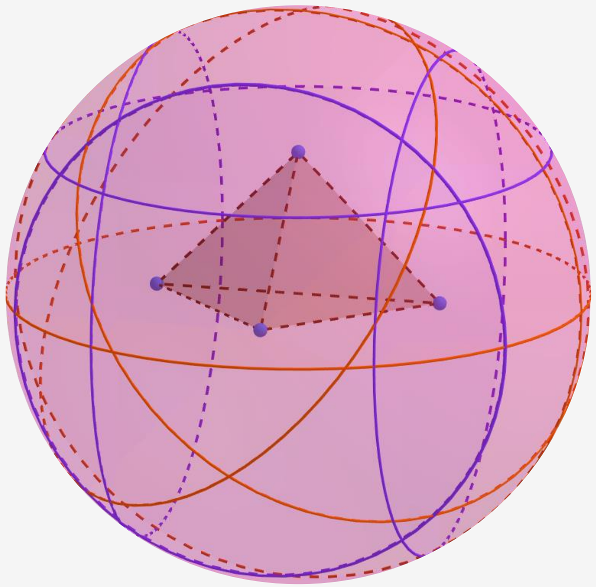

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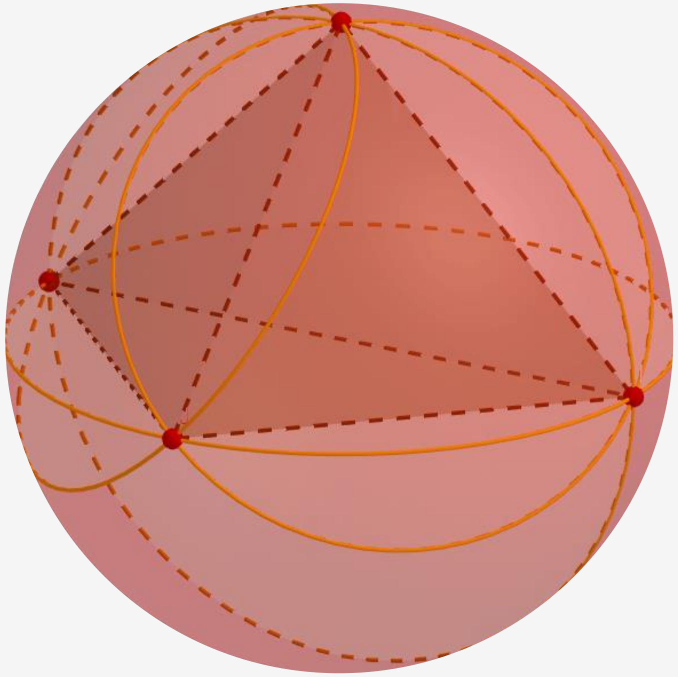
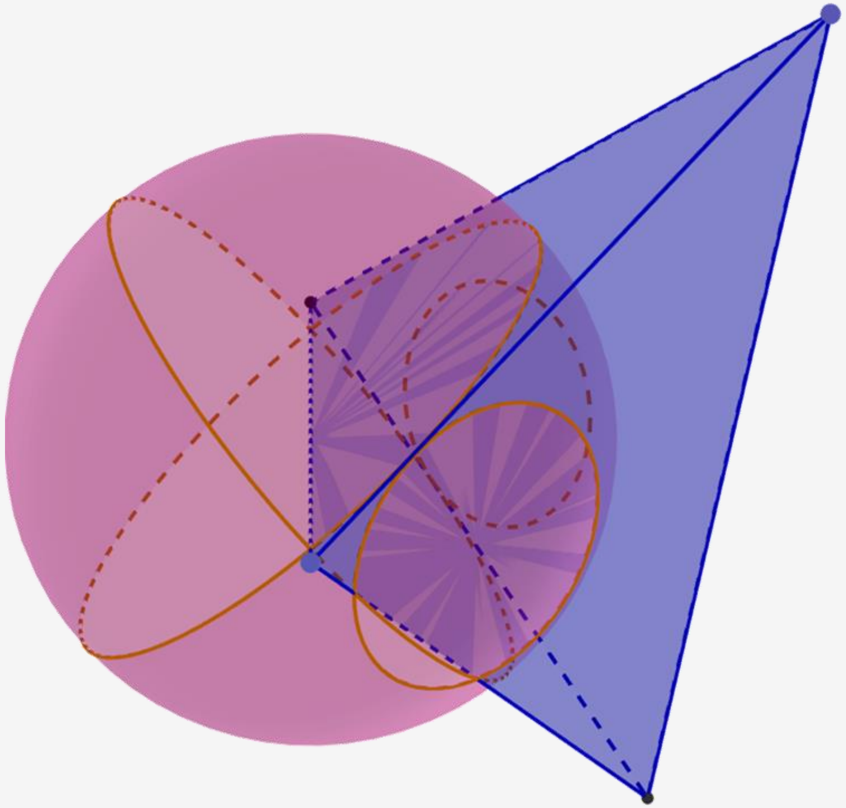


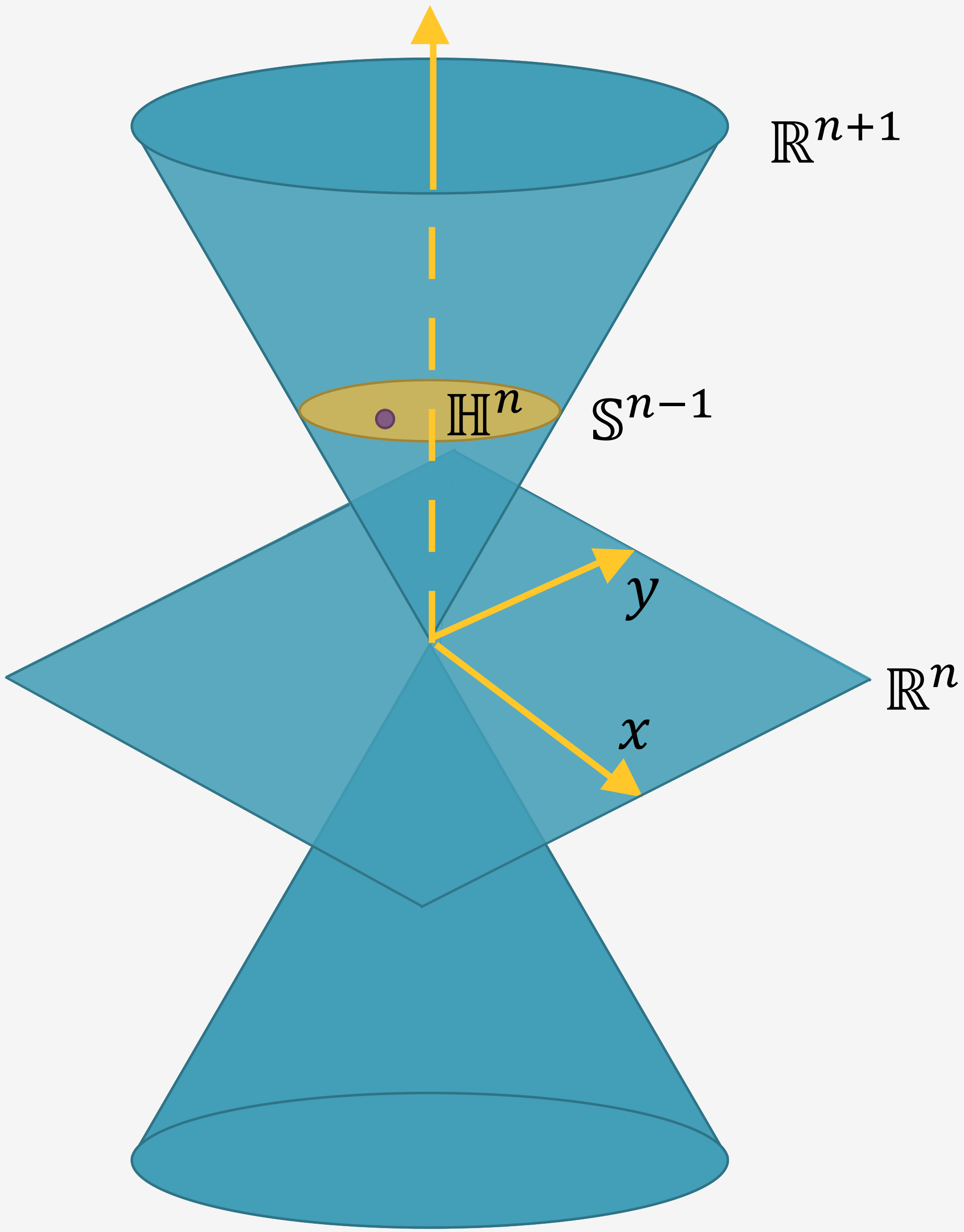


Bao & Bonahon ('02);
Bowers, Bowers, Pratt ('18)

THANK YOU!

Lorentz Space

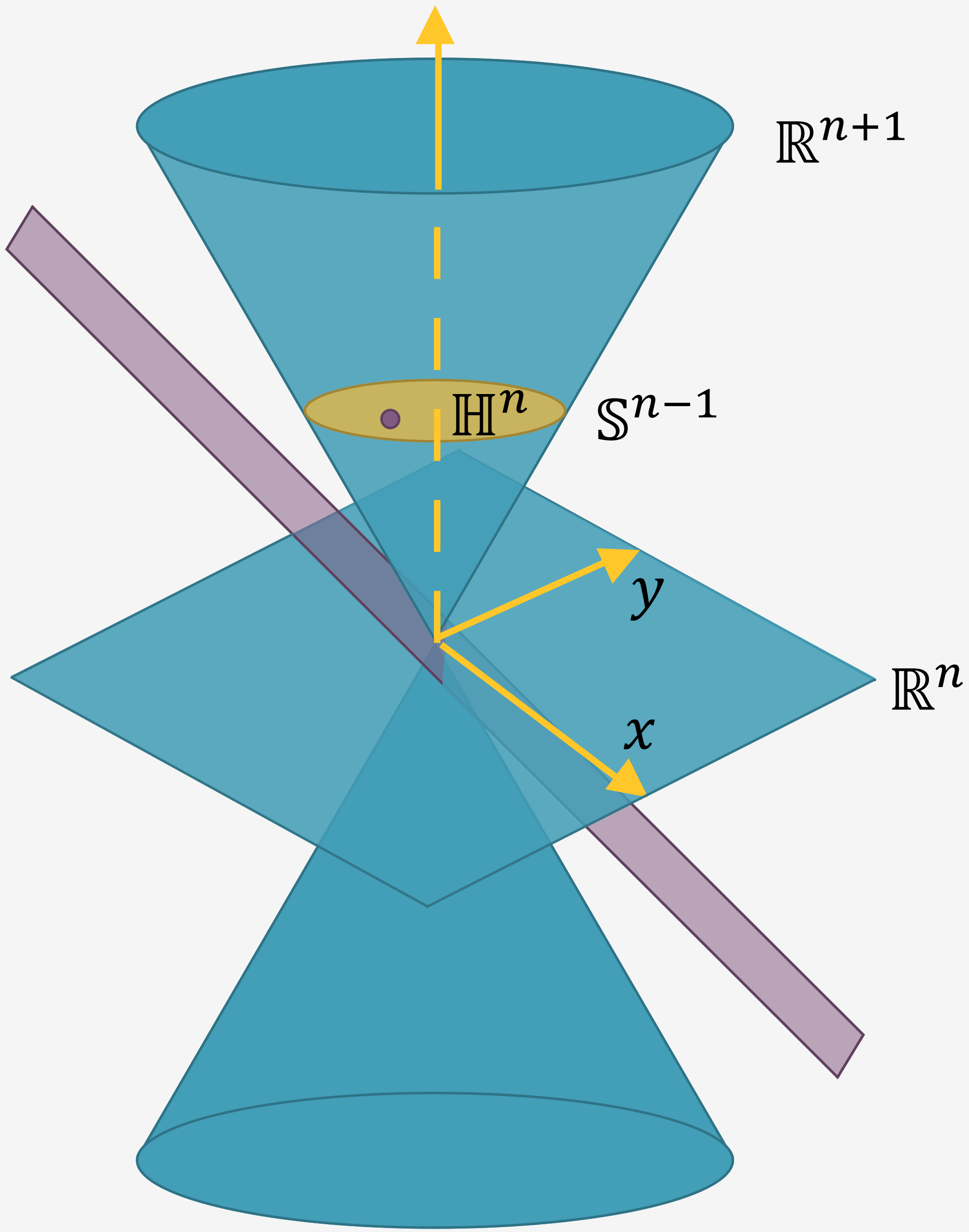
Hyperbolic Points in $\mathbb{H}^n$



Lorentz Space

Hyperbolic Points in $\mathbb{H}^n \longleftrightarrow$

Space-like Lorentz subspaces

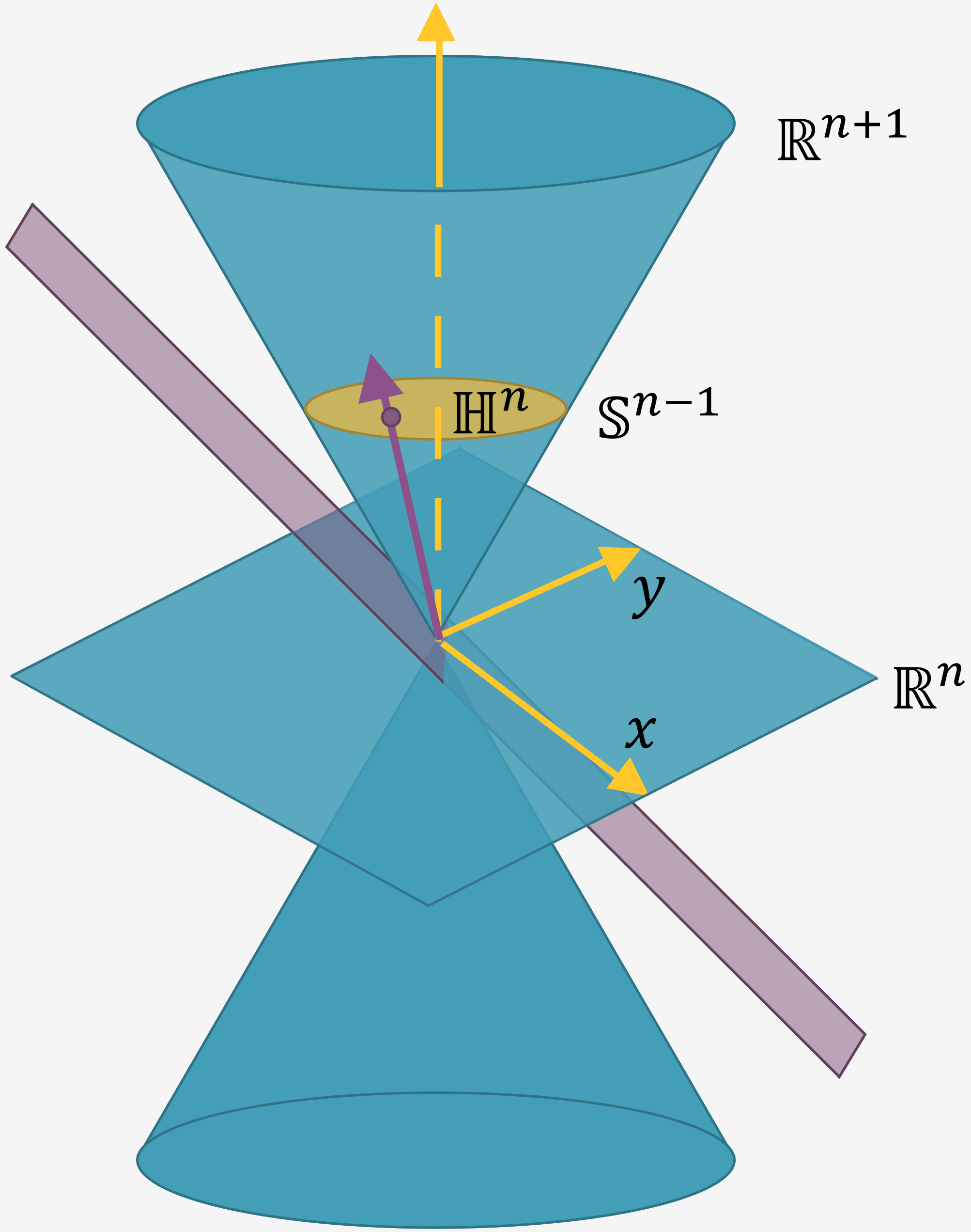


Lorentz Space

Hyperbolic Points in $\mathbb{H}^n$ ⟷

Space-like Lorentz subspaces ⟷

Time-Like Lorentz unit vectors



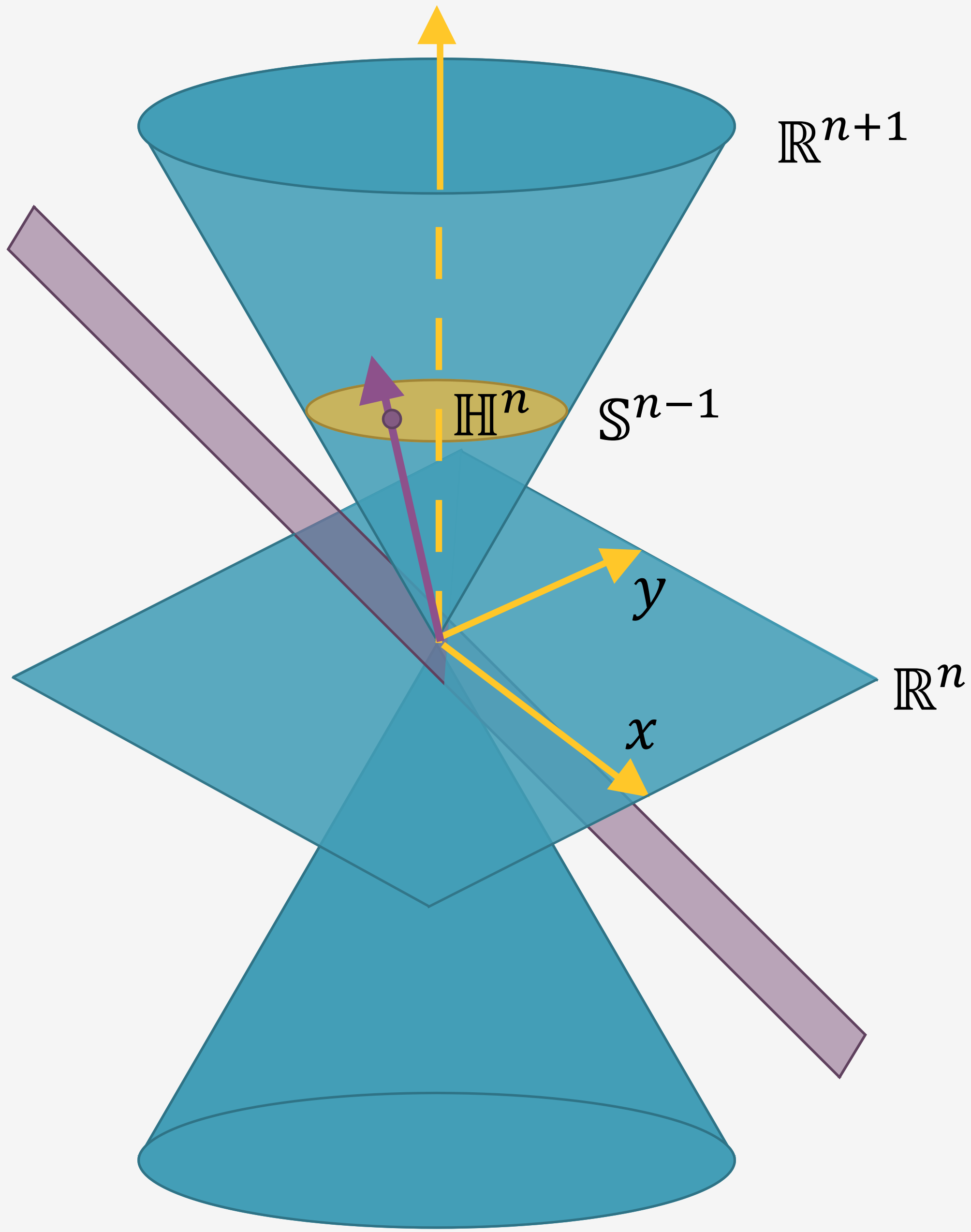
Lorentz Space

Hyperbolic Points in $\mathbb{H}^n \longleftrightarrow$

Space-like Lorentz subspaces $\longleftrightarrow$

Time-Like Lorentz unit vectors

Fact: Lorentz inner product corresponds to hyperbolic distance



New Rigidity Statement—Points

- **Ideal Points:** Only need $|p_1, p_2, p_3, p_\alpha| = |p'_1, p'_2, p'_3, p'_\alpha|$ for chosen p_1, p_2, p_3 in independent subcollection.

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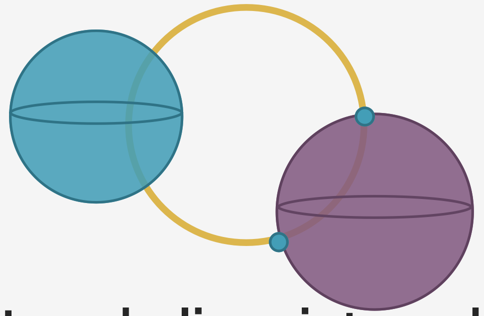

- **Hyperbolic points and hyperplanes:** Independence?

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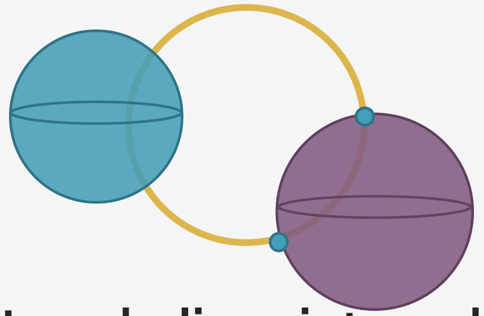

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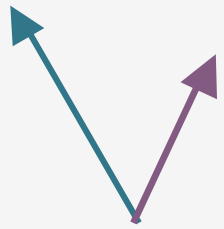

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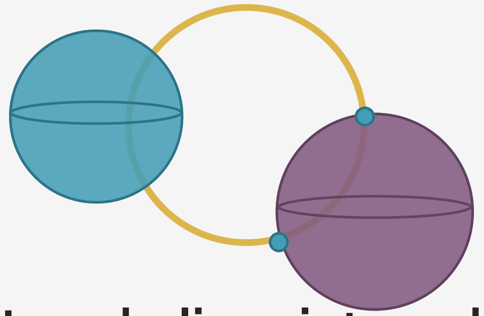

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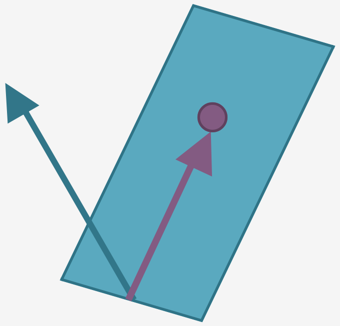
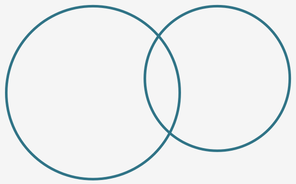


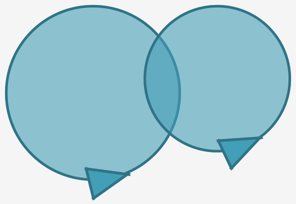
6 CASES



Inversive Distance Primer

Inversive Distance Primer

6 CASES



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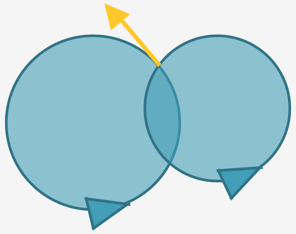
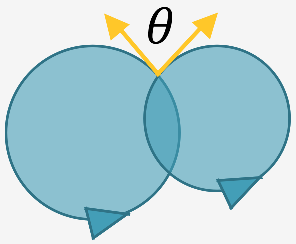


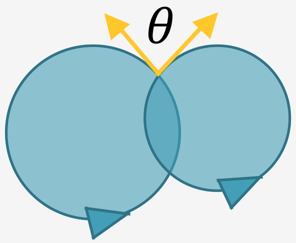
Inversive Distance Primer

6 CASES



Inversive Distance Primer

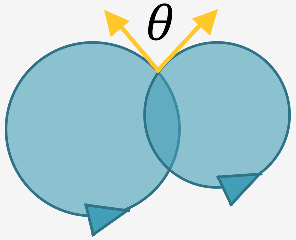
6 CASES



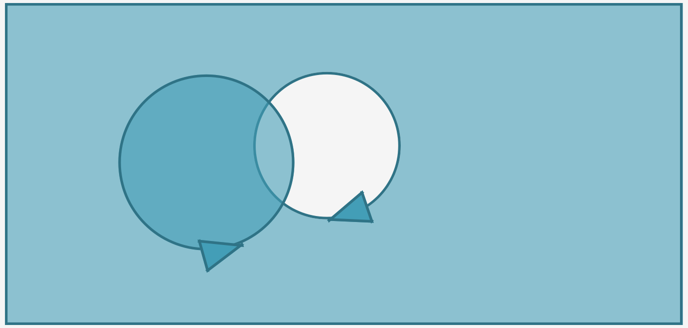
$$0 \le (C, C') = \cos(\theta) < 1$$

Inversive Distance Primer

6 CASES

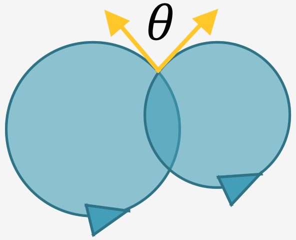


$0 \leq (C, C') = \cos(\theta) < 1$



Inversive Distance Primer

6 CASES



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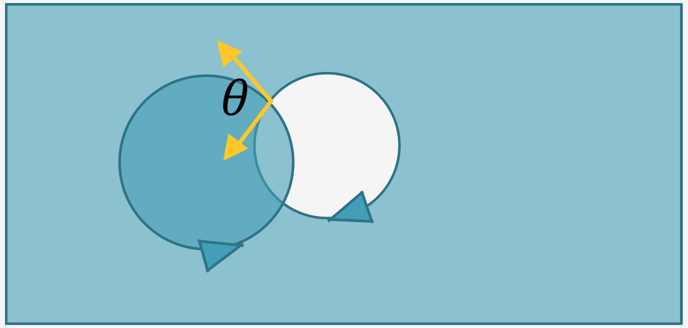
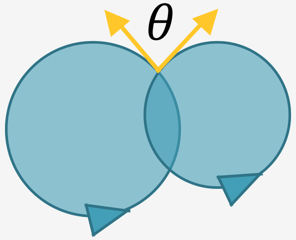
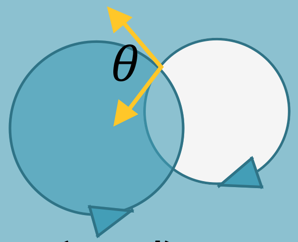


Inversive Distance Primer

6 CASES



$0 \leq (C, C') = \cos(\theta) < 1$

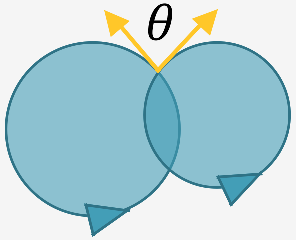


$-1 < (C, C') = \cos(\theta) \leq 0$

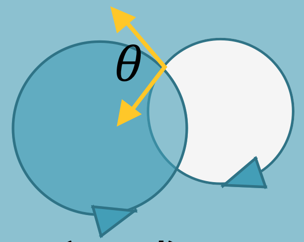
Inversive Distance Primer

Inversive Distance Primer

6 CASES

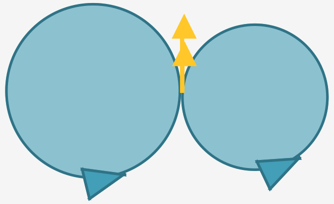


$0 \leq (C, C') = \cos(\theta) < 1$



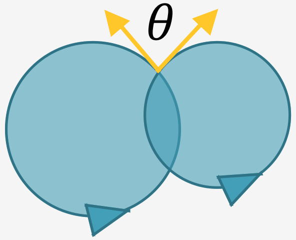
$-1 < (C, C') = \cos(\theta) \leq 0$

$\theta = \pi$



Inversive Distance Primer

6 CASES



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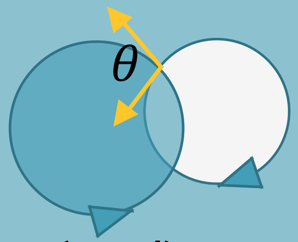

$-1 < (C, C') = \cos(\theta) \leq 0$

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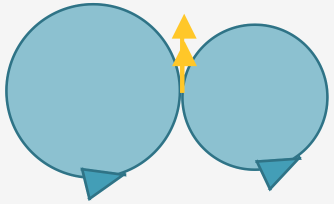
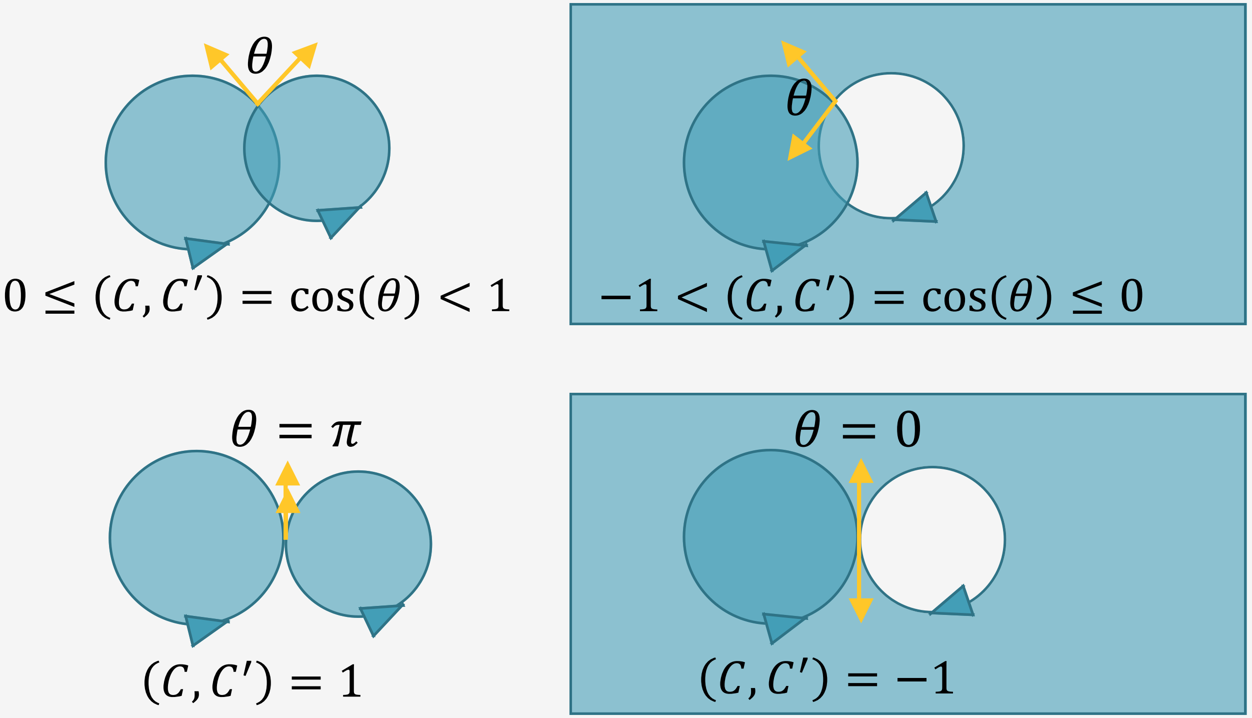


$(C, C') = 1$

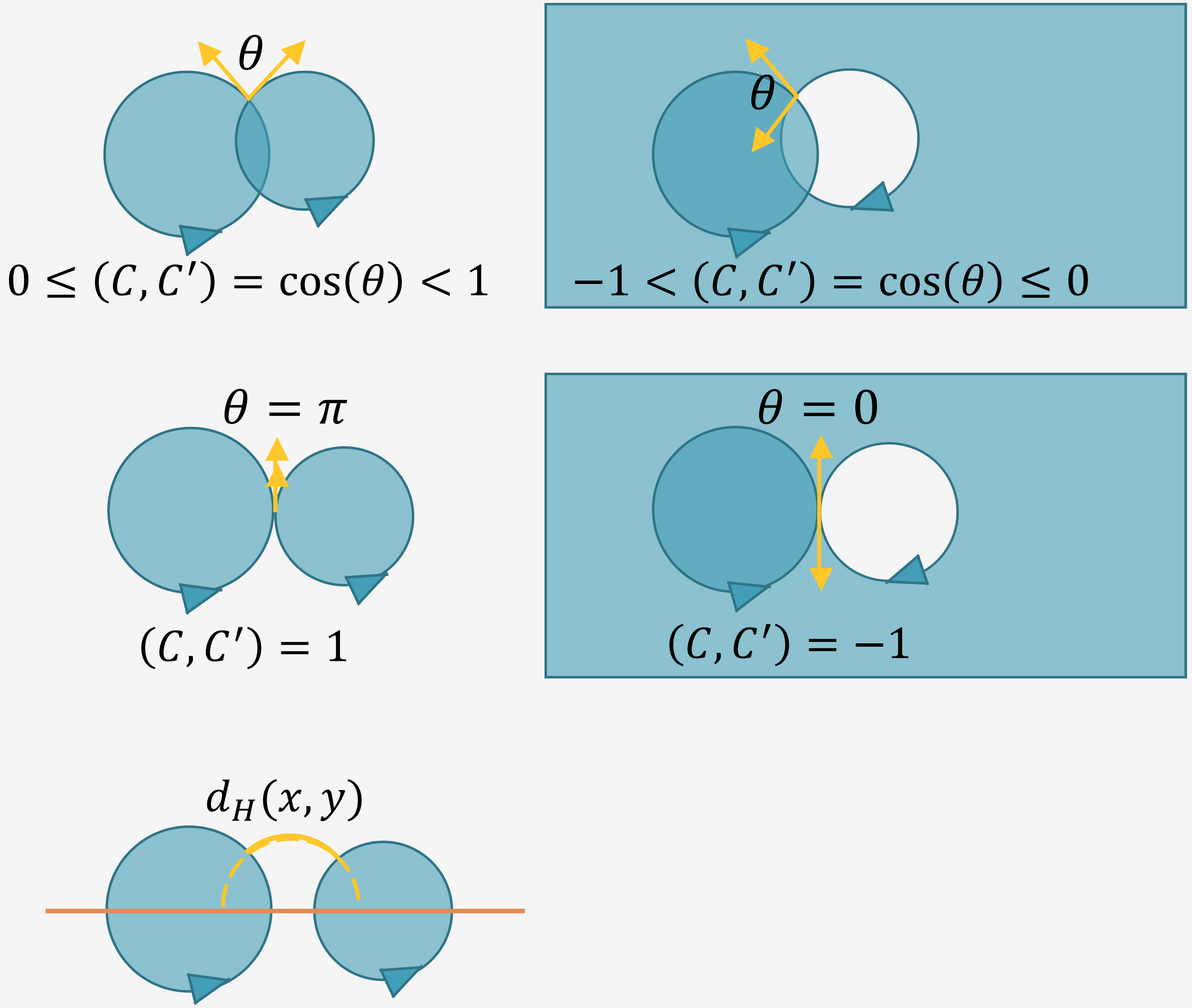
6 CASES



Inversive Distance Primer

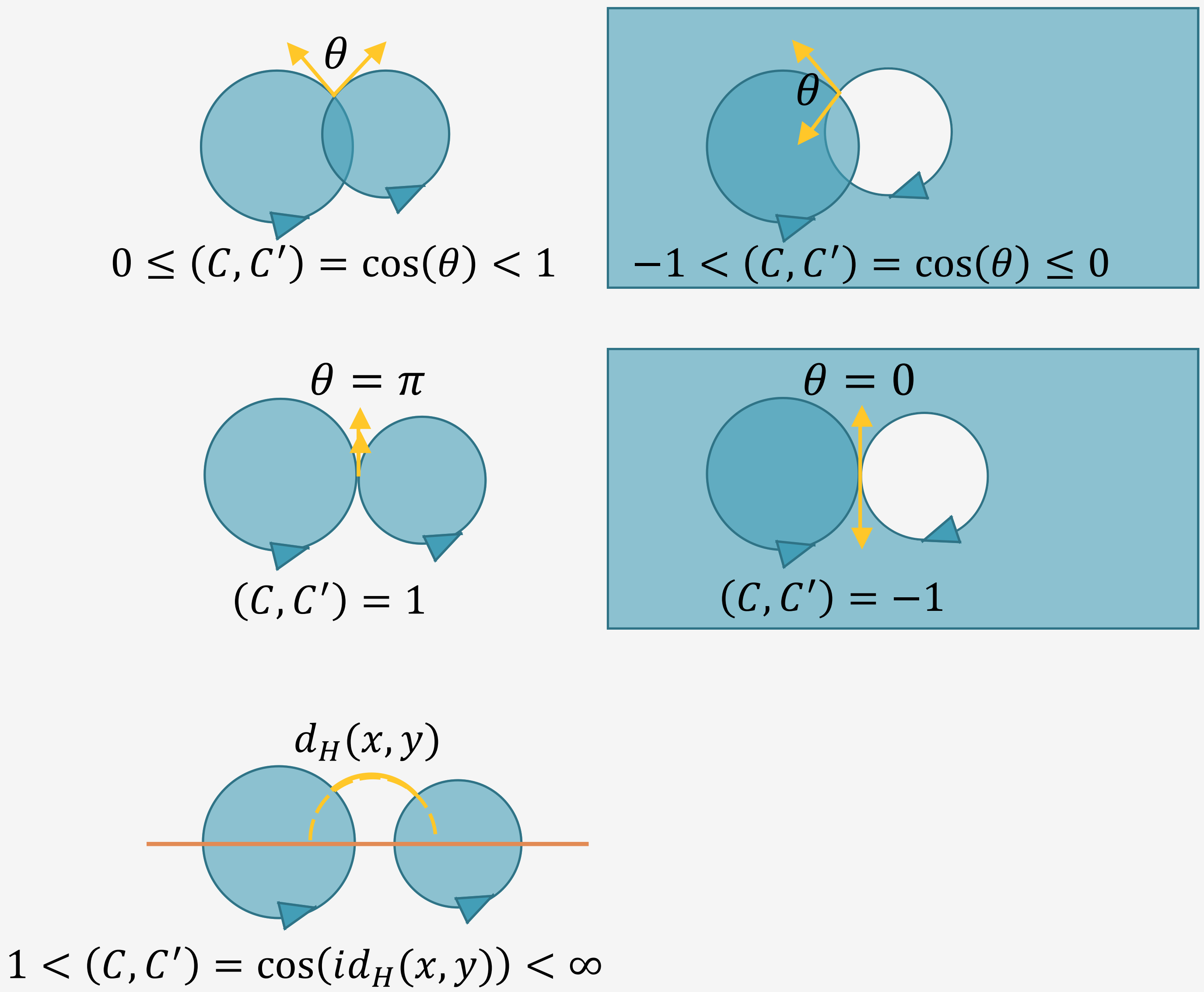
Inversive Distance Primer

6 CASES



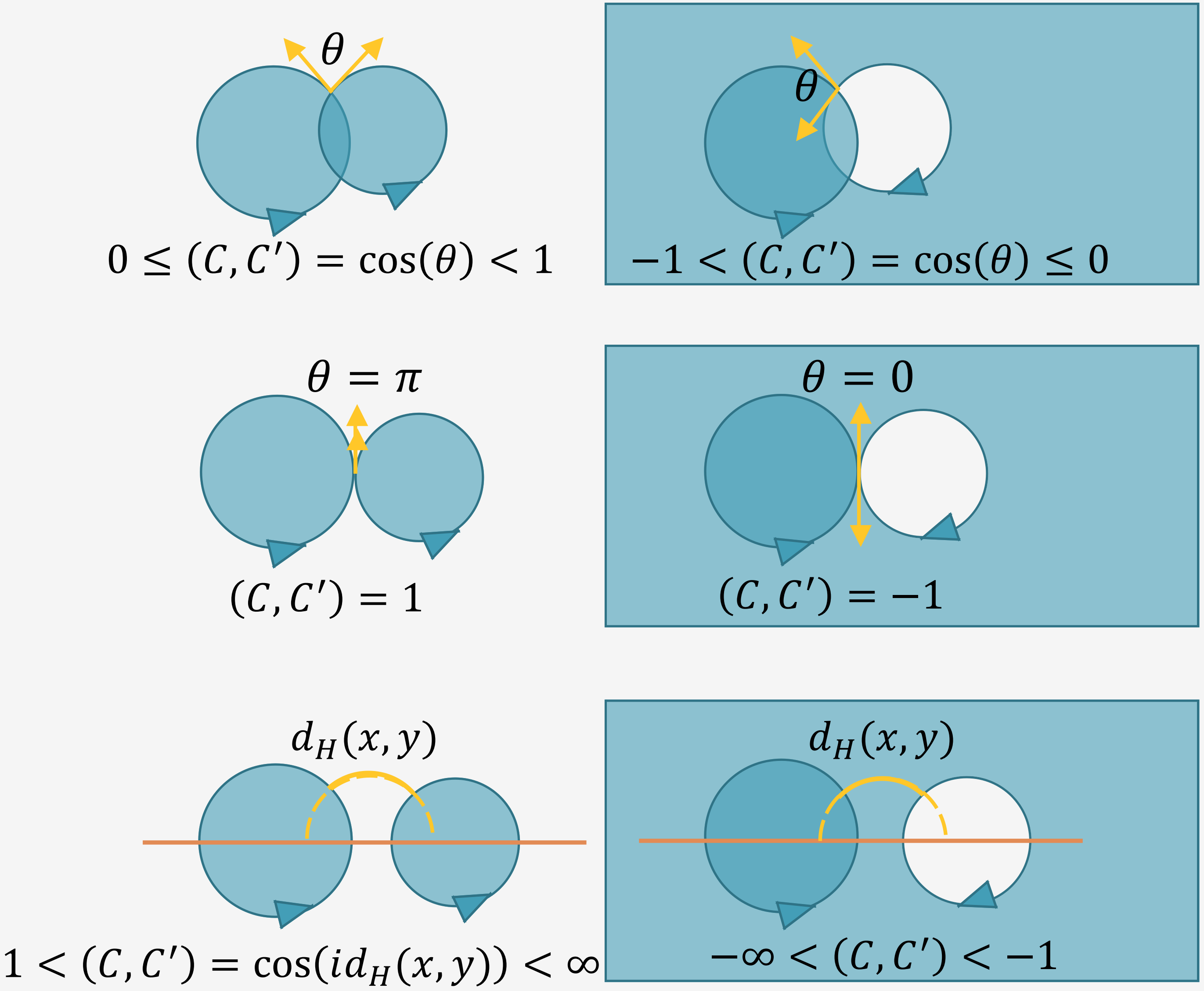
Inversive Distance Primer

6 CASES

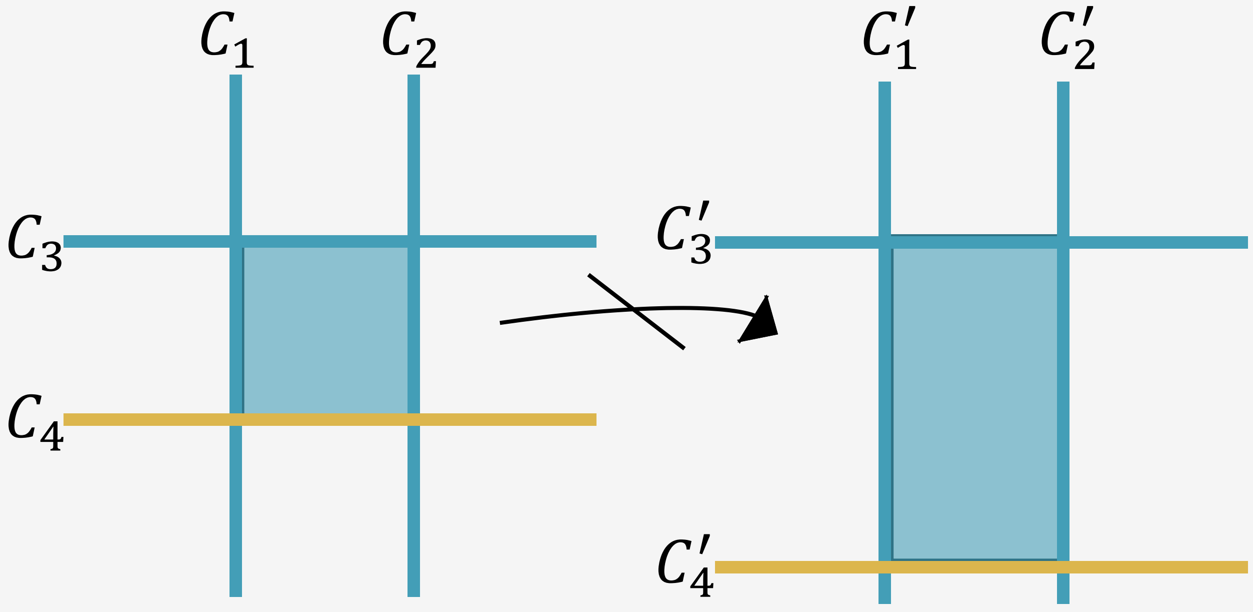


Inversive Distance Primer

6 CASES

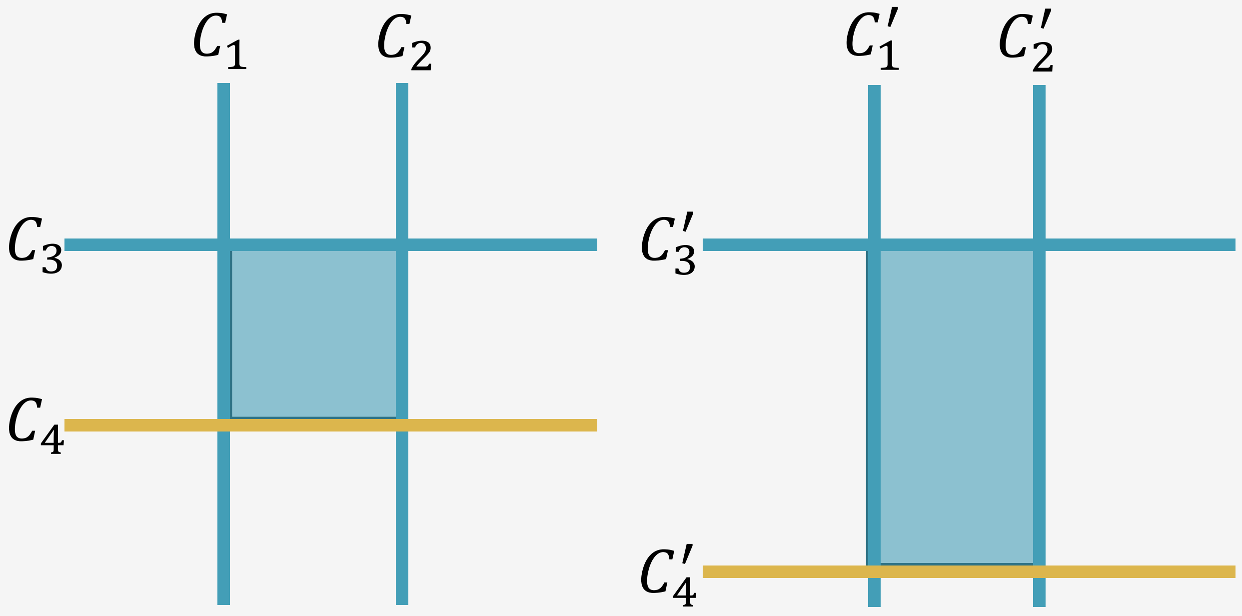


Crane & Short's Condition



$$\nexists\, f \in \text{Möb}\left(\hat{\mathbb{C}}\right)$$

Crane & Short's Condition



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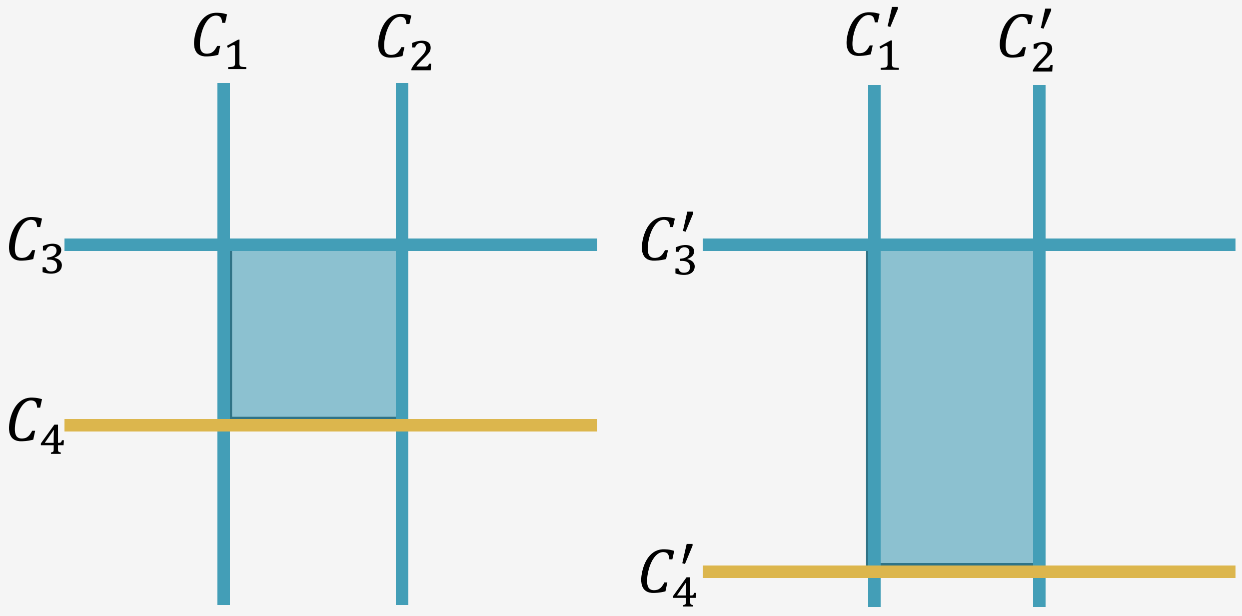
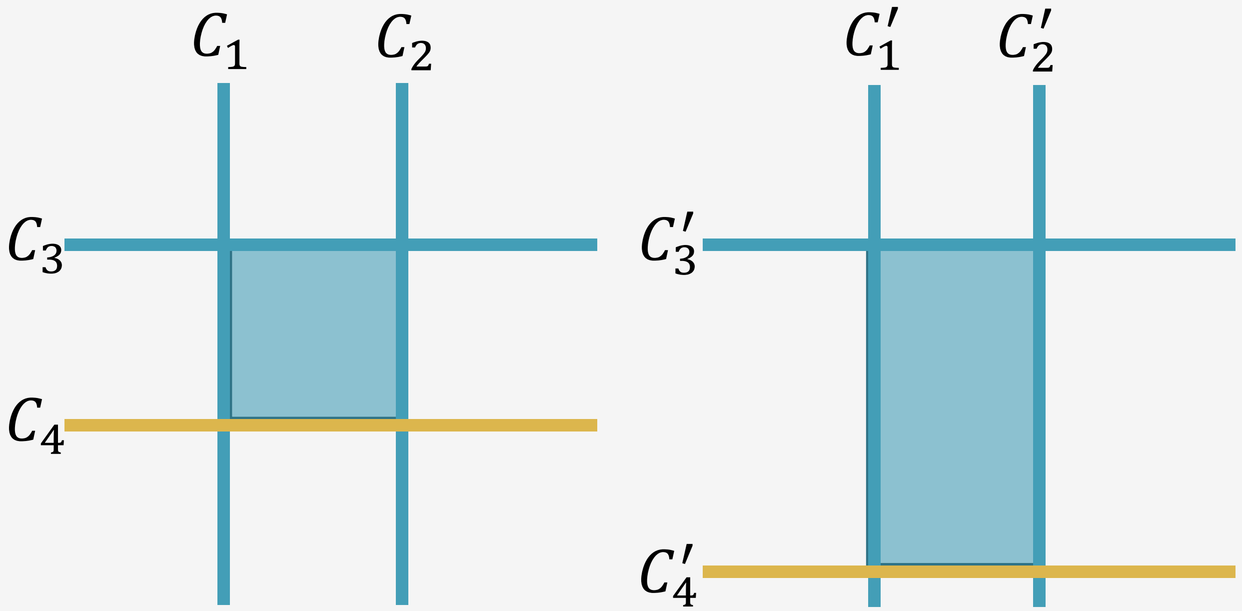


Point ∞ common to all circles

Crane & Short's Condition



Point ∞ common to all circles $\longleftrightarrow$ All space-like vectors lie on common light-like space